# New Interpretation of the Scalar Product in Hilbert Space

Y. Aharonov
*Physics Department, Tel Aviv University, Ramat Aviv, Israel, and Physics Department, University of South Carolina, Columbia, South Carolina 29208*

and

David Z. Albert
*Physics Department, The Rockefeller University, New York, New York 10021*

and

C. K. Au
*Department of Physics and Astronomy, University of South Carolina, Columbia, South Carolina 29208*
(Received 24 August 1981)

The inner product of two states in Hilbert space is interpreted in terms of a relationship between two or more distinct physical systems. This point of view suggests a generalized notion of measurement, within which the familiar types of measurement occur as special cases. A variety of novel measuring procedures are described, which relate overcomplete sets of states for a given system to complete sets for some larger system (of which the smaller system forms a part).

PACS numbers: 03.65.Ca, 03.65.Bz

The absolute square of the inner product $|\langle\psi|\varphi\rangle|^2$ of two quantum-mechanical state vectors is traditionally interpreted as the probability that a system in the state $|\varphi\rangle$ will be found to be in the state $|\psi\rangle$ if it is subjected to measurements of some complete set of observables of which $|\psi\rangle$ is an eigenstate. This Letter presents another interpretation, in which $|\langle\psi|\varphi\rangle|^2$ represents a relation between two or more distinct quantum-mechanical systems. The new interpretation carries with it a richer and more general notion of measurement, from which the familiar kinds of measurement emerge as a special case. It has a number of other advantages as well, as we shall describe below.

Let us first indicate the basic idea as it applies to a simple physical system. Consider a system of two particles (in one dimension), one of which is described by the one-particle state $|\varphi\rangle$ and the other by $|\psi^*\rangle$ (that is, by the state $\langle x|\psi\rangle^*$ in the coordinate space representation, which we have chosen for reasons which will presently be clear). Suppose that measurements of the two commuting observables $x_1-x_2$ and $p_1+p_2$ (where $x_1$, $x_2$, $p_1$, and $p_2$ are the positions and momenta of particles 1 and 2, respectively) are carried out on the system. The probability density that these measurements will yield

$$x_1-x_2=0 \quad\text{and}\quad p_1+p_2=0 \tag{1}$$

is given by

$$\begin{aligned}|\langle x_1-x_2=0,p_1+p_2=0|\varphi\cdot\psi^*\rangle|^2 &= \left|(2\pi)^{-1/2}\int_{-\infty}^{\infty} dx_1\,dx_2\,\varphi(x_1)\psi^*(x_2)\,\delta(x_1-x_2)\right|^2\\ &= \left|(2\pi)^{-1/2}\int_{-\infty}^{\infty}\varphi(x)\psi^*(x)\,dx\right|^2=(2\pi)^{-1}|\langle\psi|\varphi\rangle|^2,\end{aligned} \tag{2}$$

where $|\varphi\cdot\psi^*\rangle\equiv|\varphi\rangle|\psi^*\rangle$ and $\varphi(x)\equiv\langle x|\varphi\rangle$, etc.

This is an interesting result in a number of respects. First, we have found that $(2\pi)^{-1}|\langle\psi|\varphi\rangle|^2$ is the probability that one particle in the state $|\varphi\rangle$ and another in the state $|\psi^*\rangle$ have the same positions and opposite momenta. Since $|\psi^*\rangle$ is a time-reversed (i.e., momentum-reversed) version of $|\psi\rangle$, however, $(2\pi)^{-1}|\langle\psi|\varphi\rangle|^2$ can be understood as a propensity of two particles, one in the state $|\varphi\rangle$ and the other in the state $|\psi\rangle$, to have the same positions and the *same* momenta (such a propensity cannot be directly measured, of course, since $[p_1-p_2,x_1-x_2]\neq 0$). Thus, whereas in the traditional view, the inner product gives the projection of one state on another in Hilbert space, it appears in (2) as a measure of the proximity of $|\varphi\rangle$ to $|\psi\rangle$ in *phase* space. Second, the probability appearing in (2) is not $|\langle\psi|\varphi\rangle|^2$ itself, but rather $(2\pi)^{-1}|\langle\psi|\varphi\rangle|^2$. Two particles in the same quantum state will not necessarily coincide in space, for example, but the propensity of two particles to coincide is proportional to the absolute square of the inner product of their two states. Finally, (2) suggests a new kind of measurement process, wherein one of the two particles is initially prepared in a definite state, and is considered as part of an apparatus for measuring certain properties of the other particle. In the remainder of this Letter we will describe a few of the varieties and uses of such measurements.

To begin with, let us generalize the calculation in (2). Suppose that the state of our two-particle system is, as above, $|\varphi\rangle|\psi^*\rangle$, and that measurements of $x_1-x_2$ and $p_1+p_2$ are carried out. The probability density of finding the result

$$x_2-x_1=\beta \text{ and } p_1+p_2=\alpha \tag{3}$$

is given by

$$|\langle x_1-x_2=\beta, p_1+p_2=\alpha|\varphi\cdot\psi^*\rangle|^2=|(2\pi)^{-1/2}\int_{-\infty}^{\infty}dx_1\,dx_2\,\varphi(x_1)\psi^*(x_2)\delta(x_1-x_2-\beta)\exp[i\alpha(x_1+x_2)/2]|^2$$
$$=|(2\pi)^{-1/2}\int_{-\infty}^{\infty}dx\,\varphi(x)\psi^*(x+\beta)e^{i\alpha x}e^{i\alpha\beta/2}|^2=(2\pi)^{-1}|{}_{\psi}\langle\alpha,\beta|\psi\rangle|^2, \tag{4}$$

where[1]

$$|\alpha,\beta\rangle_\psi\equiv\exp[i(\alpha\hat{x}+\beta\hat{p})]|\psi\rangle. \tag{5}$$

Thus if the particle associated with the measuring apparatus (hereafter this particle will be called the "quantum ruler") is prepared in any state $|\psi^*\rangle$, and if the state of the other particle (hereafter called the "object") is $|\varphi\rangle$, then the probability of obtaining the results (3) is proportional to the absolute square of the inner product of $|\varphi\rangle$ with $|\alpha,\beta\rangle_\psi$ (that is, with $|\psi\rangle$ translated through a distance $\beta$ in coordinate space and $\alpha$ in momentum space).

If, for example, the quantum ruler is prepared in the position eigenstate $|\psi^*\rangle=|x=0\rangle$, then the states $|\alpha,\beta\rangle_\psi$ for all values of $\alpha$ and $\beta$ will form the complete set of all possible position eigenstates. In this case the above procedure reduces to a familiar measurement of the position of the object; that is, the probability of a given value for $\beta$ (integrating over all possible values for $\alpha$) will be exactly $|\langle x=\beta|\varphi\rangle|^2$. Similarly, if the ruler is prepared in an eigenstate of momentum, then the states $|\alpha,\beta\rangle_\psi$ will form the complete set of momentum eigenstates, and the procedure will measure the momentum of the object.

Of more interest, however, is the case where $|\psi\rangle$ is an eigenstate neither of position nor of momentum. In this case the states $|\alpha,\beta\rangle_\psi$ will form an *overcomplete* set (if $|\psi\rangle$ is taken to be a Gaussian in coordinate space, for example, the states $|\alpha,\beta\rangle_\psi$ are the well-known coherent states). Whatever state is chosen for $|\psi\rangle$, the measurements of $x_1-x_2$ and $p_1+p_2$ must necessarily yield *some* result, and thus it follows that

$$(2\pi)^{-1}\int_{-\infty}^{\infty}d\alpha\,d\beta\,|\alpha,\beta\rangle_{\psi}{}_{\psi}\langle\alpha,\beta|=I, \tag{6}$$

where $I$ is the identity operator. Equation (6) has traditionally been demonstrated by means of purely mathematical arguments[2]; in our analysis, on the other hand, (6) follows directly from the observation that $\alpha$ and $\beta$ can be interpreted as the results of measurements on a two-particle system. The possibility of distinguishing between the various elements of such an overcomplete set of states by means of measurements on some larger system (of which the system of interest is a subsystem) is a new and potentially very useful by-product of this approach.

What happens here is that a complete set of states for the two-particle system (the states $|x_1-x_2+\beta, p_1+p_2=\alpha\rangle$, for all $\alpha,\beta$) is projected, by fixing the state $|\psi^*\rangle$ of the quantum ruler, onto an *overcomplete* set (the states $|\alpha,\beta\rangle_\psi$) for the one-particle object. This technique can be exploited to generate an infinite variety of overcomplete sets of states for any given system, and each such set will be associated with a relation of the form of (6), which, as above, will follow trivially from the measurement interpretation. Thus, for example, we can employ two rulers rather than one, and here the measurement of three commuting observables of the three-particle states,

$$\begin{aligned} x_1-x_2&=\epsilon,\\ x_1-x_3&=\eta,\\ p_1+p_2+p_3&=\delta, \end{aligned} \tag{7}$$

will give rise to a three-parameter set of overcomplete states for the one-particle system, and to an associated relation

$$N\int_{-\infty}^{\infty}d\delta\,d\epsilon\,d\eta\,|\delta,\epsilon,\eta\rangle\langle\delta,\epsilon,\eta|=I, \tag{8}$$

where $N$ is a normalization constant.[3]

An analogous approach can be taken with discrete observables. If, for example, $L_1\cdot L_2$ is measured in a two-spin system, together with $L_1^{(z)}+L_2^{(z)}$, the set of possible results can be used to parametrize an overcomplete set of states for one of the spins (the object) in terms of its projection on the other (the ruler).

In every case relations of the form (6) and (8) will emerge, and frequently theorems (such as are familiar for the coherent states) which assure the representability of any operator in terms of its diagonal matrix elements in an overcomplete basis will follow in a very natural way from our analysis.[4,5]

These issues and others will be the subjects of a forthcoming publication. The purpose of the present note is to introduce an alternative interpretation of the inner product as a relation between two or more distinct quantum-mechanical systems, and to indicate how this interpretation naturally lends itself to a new and more general notion of measurement, wherein various quantum-mechanical systems are compared directly to one another, rather than to a classical "measuring rod."

This work was supported by the National Science Foundation under Grant No. ISP-80-11451.

[1]Where $\hat{x}$ and $\hat{p}$ are operators. Note that in the coordinate space representation

$$\exp[i(\alpha\hat{x}+\beta\hat{p})]=\exp(i\alpha x)\exp(i\beta p)\exp(i\tfrac{1}{2}\alpha\beta).$$

[2]See J. R. Klauder, J. Math. Phys. (N.Y.) 4, 1055, 1058 (1963).

[3]The states $|\delta,\epsilon,\eta\rangle$ will in general take the form of products of two wave functions displaced by different amounts in phase space, viz,

$$\psi(x+\epsilon)\varphi(x+\eta)e^{i\delta x}.$$

The interpretation of such products as one-particle states (i.e., as eigenstates of one-particle observables) is cumbersome, at best, within the traditional point of view. In our analysis, on the other hand, they emerge in a very natural way.

[4]R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[5]J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. (N.Y.) 6, 734 (1965).