Cherenkov radiation of superluminal particles

Daniel Rohrlich[1,]* and Yakir Aharonov[2]
[1]*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*School of Physics and Astronomy, Tel Aviv University, Ramat Aviv, 69978 Tel Aviv, Israel and Department of Physics, University of South Carolina, Columbia, South Carolina 29208*
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Any charged particle moving faster than light through a medium emits Cherenkov radiation. We show that charged particles moving faster than light through the *vacuum* emit Cherenkov radiation. How can a particle move faster than light? The *weak* speed of a charged particle can exceed the speed of light. By definition, the weak velocity $\langle \mathbf{v} \rangle_w$ is $\langle \Psi_{fin}|\mathbf{v}|\Psi_{in}\rangle / \langle \Psi_{fin}|\Psi_{in}\rangle$, where $\mathbf{v}$ is the velocity operator and $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ are, respectively, the states of a particle before and after a velocity measurement. We discuss the consistency of weak values and show that superluminal weak speed is consistent with relativistic causality.

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I. INTRODUCTION

In quantum mechanics, it is axiomatic that the only allowed values of an observable are its eigenvalues. With these allowed values come, in turn, allowed interpretations. For example, a quantum particle can tunnel through a potential-energy barrier greater than its total energy. Can it have negative kinetic energy? The axiomatic answer is "No, the eigenvalues of kinetic energy are all positive." This answer does not allow us an intuitive interpretation of quantum tunneling as a negative kinetic-energy phenomenon. But we can go beyond the axiomatic answer to define the *weak* value $\langle A \rangle_w$ of an observable A on a system [1,2]:

$$\langle A \rangle_w = \frac{\langle \Psi_{fin}|A|\Psi_{in}\rangle}{\langle \Psi_{fin}|\Psi_{in}\rangle}. \tag{1}$$

Here $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ are, respectively, the states of the system before and after a measurement of A. (Just as we can preselect $|\Psi_{in}\rangle$, we can postselect $|\Psi_{fin}\rangle$; thus we measure A on a preselected and postselected ensemble.) Weak values are measurable. If the measurement interaction is weak enough [1,2], measurements on a preselected and postselected ensemble yield the weak value $\langle A \rangle_w$, and $\langle A \rangle_w$ need *not* be an eigenvalue. Indeed, it need not be any classically allowed value. The weak kinetic energy of a tunneling particle is *negative* [3]. Weak values allow many new interpretations, in addition to negative kinetic energy. Here we show that the weak speed of a particle can exceed the speed of light, and we discuss the consistency of weak values.

We will begin by showing how the weak speed of a charged particle can exceed the speed of light *in vacuo*. Such behavior seems completely inconsistent with the laws of physics. But we then compute the electromagnetic field of the particle and find that it corresponds to Cherenkov radiation: like any charged particle moving faster than light through a medium, a superluminal particle emits Cherenkov radiation. Finally, we prove that superluminal weak speed does not contradict relativistic causality. Weak speed illustrates the general principle that all values measured on a preselected and postselected ensemble are consistent.

II. QUANTUM WALK

Consider a particle constrained to move along the z axis. As a model Hamiltonian for our particle, we take $H = p_z v_z$, where $p_z = -i\hbar\,\partial/\partial z$ and v_z acts on an internal Hilbert space of the particle:

$$v_z = \frac{c}{N}\sum_{i=1}^{N} \sigma_z^{(i)}. \tag{2}$$

The Pauli matrices operate on the internal Hilbert space. (They do not represent spin—the particle has no electric or magnetic dipole moment.) The eigenvalues of v_z are $-c$, $-c+2c/N, \ldots, c-2c/N, c$, where c is the speed of light. The particle moves with velocity v_z in the z direction,

$$\dot{x} = [x,H]/i\hbar = 0, \quad \dot{y} = [y,H]/i\hbar = 0, \quad \dot{z} = [z,H]/i\hbar = v_z; \tag{3}$$

hence the change in position z is a measure of v_z.

If the only allowed values of v_z are its eigenvalues, the speed of the particle cannot exceed the speed of light. But consider the following weak measurement of v_z. We preselect the particle in an initial state $|\Psi_{in}\rangle\Phi(\mathbf{x},0)$, where $\Phi(\mathbf{x},0)$ represents a particle approximately localized at $\mathbf{x} = (x,y,z) = 0$,

$$\Phi(\mathbf{x},0) = (\epsilon^2\pi)^{-3/4} e^{-\mathbf{x}^2/2\epsilon^2}, \tag{4}$$

and postselect a final state $|\Psi_{fin}\rangle$. For $|\Psi_{in}\rangle$ and $|\Psi_{fin}\rangle$ we choose

$$|\Psi_{in}\rangle = 2^{-N/2} \otimes_{i=1}^{N} (|\uparrow_i\rangle + |\downarrow_i\rangle),$$

$$|\Psi_{fin}\rangle = \otimes_{i=1}^{N} (\alpha_\uparrow |\uparrow_i\rangle + \alpha_\downarrow |\downarrow_i\rangle), \tag{5}$$

with $\alpha_\uparrow$ and $\alpha_\downarrow$ real and $\alpha_\uparrow^2 + \alpha_\downarrow^2 = 1$. Our chances of postselecting the state $|\Psi_{fin}\rangle$ may be very small, but if we repeat the experiment again and again, eventually we will postselect $|\Psi_{fin}\rangle$. Thus $\Phi(\mathbf{x},t)$ is

*Email address: rohrlich@wicc.weizmann.ac.il

$$\Phi(\mathbf{x},t)=\langle\Psi_{fin}|e^{-ip_zv_zt/\hbar}|\Psi_{in}\rangle\Phi(\mathbf{x},0), \tag{6}$$

up to normalization. For short enough times t, we can expand the exponent:

$$\begin{aligned}\Phi(\mathbf{x},t)&\approx\langle\Psi_{fin}|1-ip_zv_zt/\hbar|\Psi_{in}\rangle\Phi(\mathbf{x},0)\\&=\langle\Psi_{fin}|1-ip_z\langle v_z\rangle_w t/\hbar|\Psi_{in}\rangle\Phi(\mathbf{x},0)\\&\approx\langle\Psi_{fin}|e^{-ip_z\langle v_z\rangle_w t/\hbar}|\Psi_{in}\rangle\Phi(\mathbf{x},0)\\&=\langle\Psi_{fin}|\Psi_{in}\rangle\Phi(x,y,z-\langle v_z\rangle_w t,0).\end{aligned} \tag{7}$$

Thus at time t the particle is displaced by $\langle v_z\rangle_w t$ along the z axis. Note that the weak value of v_z,

$$\langle v_z\rangle_w=\frac{\langle\Psi_{fin}|v_z|\Psi_{in}\rangle}{\langle\Psi_{fin}|\Psi_{in}\rangle}=\frac{\alpha_\uparrow-\alpha_\downarrow}{\alpha_\uparrow+\alpha_\downarrow}c, \tag{8}$$

exceeds c in magnitude if $\alpha_\uparrow\alpha_\downarrow$ is negative. Thus the weak speed of the particle could be superluminal.

This result is surprising enough to merit a second derivation. We can rewrite Eq. (6) by evaluating the exponent exactly:

$$\begin{aligned}\Phi(\mathbf{x},t)&=2^{-N/2}(\alpha_\uparrow e^{-ip_zct/N\hbar}+\alpha_\downarrow e^{ip_zct/N\hbar})^N\Phi(\mathbf{x},0)\\&=2^{-N/2}\sum_{n=0}^{N}\alpha_\uparrow^n\alpha_\downarrow^{N-n}\frac{N!}{n!(N-n)!}\\&\quad\times e^{-i(2n-N)p_zct/N\hbar}\Phi(\mathbf{x},0).\end{aligned} \tag{9}$$

Equation (9) represents a superposition of many displacements of the particle. Applying the binomial theorem, we find that $\Phi(\mathbf{x},t)$ is a superposition of $\Phi(\mathbf{x},0)$ displaced along the z axis by at most ct in either direction. So how can Eq. (6) represent a particle displaced by $\langle v_z\rangle_w t$ if $\langle v_z\rangle_w t$ is out of this range? Here is the surprise. Apparently the displaced states interfere, *constructively* for $z\approx\langle v_z\rangle_w t$ and *destructively* for other values of z. Indeed, we can verify this interference. Since

$$\begin{aligned}&\alpha_\uparrow e^{-ip_zct/N\hbar}+\alpha_\downarrow e^{ip_zct/N\hbar}\\&\quad\approx\alpha_\uparrow(1-ip_zct/N\hbar)+\alpha_\downarrow(1+ip_zct/N\hbar)\\&\quad=(\alpha_\uparrow+\alpha_\downarrow)-(\alpha_\uparrow-\alpha_\downarrow)ip_zct/N\hbar\\&\quad=(\alpha_\uparrow+\alpha_\downarrow)(1-ip_z\langle v_z\rangle_w t/N\hbar)\end{aligned} \tag{10}$$

and

$$\lim_{N\to\infty}(1-ip_z\langle v_z\rangle_w t/N\hbar)^N=e^{-ip_z\langle v_z\rangle_w t/\hbar}, \tag{11}$$

we find that, for large enough N, Eq. (9) does indeed imply Eq. (7).

Mathematically, Eq. (9) does not look like Eq. (7). Equation (9) corresponds to a superposition of waves $e^{-ip_zv_zt/\hbar}$, where $v_z=-c,-c+2c/N,\ldots,c-2c/N,c$. If $e^{-ip_z\langle v_z\rangle_w t/\hbar}$ is not one of these waves, how can we obtain it by superposing them? Physically, Eq. (9) is analogous to a random walk. We can generate a random walk in one dimension by tossing a coin. In Eq. (9), we toss a quantum coin—a spin—to generate a quantum random walk [4]. If the coefficients $\alpha_\uparrow$ and $\alpha_\downarrow$ in Eq. (9) were probabilities, the expansion of Eq. (9) would generate a classical random walk; each term in the expansion would represent a possible random walk, with a coefficient equal to its probability. A classical random walk of N steps yields a typical displacement of $\sqrt{N}$ steps, and never more than N. But the coefficients $\alpha_\uparrow$ and $\alpha_\downarrow$ are probability amplitudes; the quantum random walk superposes all possible classical random walks and yields arbitrary displacements.

III. CONSISTENCY OF WEAK VALUES

We have derived Eq. (7) in two ways, but we have not explained how such a surprising result as superluminal speed could coexist with relativistic causality (i.e., the constraint $-c\leqslant v_z\leqslant c$ that applies to v_z and its eigenvalues). The explanation is that superluminal speed depends on apparent ‘‘errors’’ of measurement. A hint of this dependence appears already in Eq. (4), where we define the initial wave function $\Phi(\mathbf{x},0)$ of the particle to be a Gaussian with an uncertainty in position of about ϵ. If ϵ vanished, $\Phi(\mathbf{x},0)$ would be a δ function of position and no superluminal behavior could emerge from Eq. (9); there would be no tails on the wave function that could interfere constructively for $z\approx\langle v_z\rangle_w t$. However, ϵ does not vanish, our initial and final measurements are uncertain, and we can obtain, ‘‘by error,’’ a displacement corresponding to superluminal speed. Thus the weak value emerges only if it could be an error; yet the weak value does not *seem* to be an error. On the contrary, whenever our preselections and postselections [which are independent of $\Phi(\mathbf{x},0)$] yield the weak value $\langle v_z\rangle_w$, measured values of the displacement of the particle over a time t cluster about $\langle v_z\rangle_w t$.

We can quantify the dependence of weak speed on measurement error as follows. Equations (7) and (9) agree in the limit $N\to\infty$, but let us take into account the fact that N is finite. To do so, we define a function $f(1/N)=(1+s/N)^N$ with s constant, and expand $f(1/N)$ in a Taylor-series expansion around $f(0)$:

$$f(1/N)=f(0)+f'(0)/N+f''(0)/2N^2+\cdots, \tag{12}$$

where $f(0)=\lim_{N\to\infty}f(1/N)$, etc. We obtain

$$\left(1+\frac{s}{N}\right)^N=e^s\left(1-\frac{s^2}{2N}+\frac{3s^4+8s^3}{24N^2}+\cdots\right). \tag{13}$$

Hence Eqs. (9) and (10) imply

$$\Phi(\mathbf{x},t)=e^{-ip_z\langle v_z\rangle_w t/\hbar}\left[1+\frac{p_z^2\langle v_z\rangle_w^2t^2}{2N\hbar^2}+O\left(\frac{1}{N^2}\right)\right]\Phi(\mathbf{x},0), \tag{14}$$

up to normalization. The exponential factor outside the brackets displaces $\Phi(\mathbf{x},0)$ by $\langle v_z\rangle_w t$, but terms of order $1/N$ can change the shape of $\Phi(\mathbf{x},0)$. To make the change negligible, we require

$$1 \gg \langle v_z \rangle_w^2 t^2 / N \epsilon^2. \tag{15}$$

Equation (15) relates N to the width ϵ of $\Phi(\mathbf{x},0)$: to decrease ϵ, we increase N. As long as Eq. (15) holds, the particle will move with weak speed $\langle v_z \rangle_w$ over a time t.

Equation (15) is crucial to the consistency of weak speed. Does it seem that we get superluminal speed by playing a "game of errors" with the measuring device? Perhaps; but it is a remarkably consistent game: whenever we preselect $|\Psi_{in}\rangle$ and postselect $|\Psi_{fin}\rangle$ of Eq. (5), we get superluminal speed (up to the uncertainty that characterizes the measuring device). For this consistency to hold, the probability of postselecting $|\Psi_{fin}\rangle$ must be *smaller* than the probability of getting the superluminal speed "by error." Otherwise, when we postselect $|\Psi_{fin}\rangle$, we would most likely *not* get superluminal speed.

Let us check: on the one hand, a particle with wave function $\Phi(\mathbf{x},0)$ may be found, by error, at $z = \langle v_z \rangle_w t$ a time t later. The probability of such an error is proportional to $e^{-\langle v_z \rangle_w^2 t^2/\epsilon^2}$, which by Eq. (15) is much greater than e^{-N}. On the other hand, the probability of postselecting the state $|\Psi_{fin}\rangle$ is approximately $(\alpha_\uparrow \alpha_\downarrow + 1/2)^N$. If we compare the two probabilities and recall that $\alpha_\uparrow \alpha_\downarrow$ is negative for $\langle v_z \rangle_w > c$, we find that the probability of an error dominates the probability of postselecting $|\Psi_{fin}\rangle$.

Then why all the fuss about postselection? If we measure v_z and obtain the value $v_z > c$, how does it matter whether or not we postselect? The answer is that *only if we postselect are measured values consistent*. An example may help clarify this answer. Suppose we measure the displacement of the particle at time t with a weak measurement interaction. If we do not postselect, the most likely displacement at time t is $ct/\sqrt{N}$, because the expectation value of v_z in the state $|\Psi_{in}\rangle$ is a random walk of N steps of size c/N. Yet there is a small chance of obtaining a displacement ct. Such a value might be an error and, indeed, if we remeasure z, there is again only a small chance of measuring such a large displacement. Since each measurement hardly disturbs the particle [5], the probability that the next measurement yields a displacement ct remains small. Thus, without postselection, there is no consistency in measurement errors. Unless and until we postselect, they are just errors. With postselection, however, measurement "errors" yield a consistent pattern. Repeated weak measurements on an ensemble of particles preselected in the state $|\Psi_{in}\rangle$ and postselected in the state $|\Psi_{fin}\rangle$ yield errors consistent with the superluminal weak value.

Equations (7) and (9) show that the weak speed of a particle can consistently exceed c. We now give the particle a charge q and show that its electromagnetic field, too, is consistent with superluminal weak speed.

IV. CHERENKOV RADIATION

What is the electromagnetic field of the particle? Let us treat the scalar potential; the treatment of the vector potential is similar. To begin with, suppose that v_z is well defined, i.e., that v_z equals one of its eigenvalues. Let $V(\mathbf{x}',t;v_z)$ denote the scalar potential at $\mathbf{x}',t$ of a particle of charge q moving along the z axis with $z = v_z t$. The simplest way to obtain $V(\mathbf{x}',t;v_z)$ is via a Lorentz boost, by v_z in the z direction, of the Coulomb potential $V(\mathbf{x}',t;0)$. We obtain

$$V(\mathbf{x}',t;v_z) = q\{[(x')^2 + (y')^2](1 - v_z^2/c^2) + (z' - v_z t)^2\}^{-1/2}. \tag{16}$$

So far, $V(\mathbf{x}',t;v_z)$ represents the classical potential of a point charge moving along the axis with $z = v_z t$. But we want to treat the field as quantum mechanical. We could do so with quantum field operators, but the treatment would be unnecessarily complicated. Instead, let us write down an effective two-particle interaction between the moving charge and a test particle. Namely, to $H = p_z v_z$, the Hamiltonian of the moving charge, we add the Hamiltonian H' of a (nonrelativistic) test particle:

$$H' = \frac{1}{2m}(\mathbf{p}' - q'\mathbf{A})^2 + q'V. \tag{17}$$

In H', the test particle has charge q', and the scalar potential is

$$V(\mathbf{x}') = q\{[(x'-x)^2 + (y'-y)^2] \times (1 - v_z^2/c^2) + (z'-z)^2\}^{-1/2}. \tag{18}$$

The vector potential has only one nonzero component, namely, A_z, which is [6]

$$A_z(\mathbf{x}') = \frac{q v_z}{c}\{[(x'-x)^2 + (y'-y)^2] \times (1 - v_z^2/c^2) + (z'-z)^2\}^{-1/2}. \tag{19}$$

Note that if we substitute $(0,0,v_z t)$ for (x,y,z), then $V(\mathbf{x}')$ equals $V(\mathbf{x}',t;v_z)$ as defined above in Eq. (16) and $A_z(\mathbf{x}')$ equals $(v_z/c)V(\mathbf{x}',t;v_z)$. The equations of motion flowing from $H + H'$ yield $(x,y,z) = (0,0,v_z t)$, together with the correct motion of the test particle due to the electromagnetic field of the moving charge. (The equation of motion for the momentum $\mathbf{p}$ of the moving charge is unphysical, but it has no measurable consequences.) Now we treat V and A_z as quantum operators and calculate their effect on the test particle. We will see that if the moving charge has weak speed $\langle v_z \rangle_w$, then $\langle v_z \rangle_w$ replaces v_z in Eqs. (18) and (19).

Namely, suppose we preselect the moving charge in the state $|\Psi_{in}\rangle \Phi(\mathbf{x},0)$ and, after a time T, postselect the state $|\Psi_{fin}\rangle$ [see Eqs. (4) and (5)]. We also prepare the test particle in a localized state $\Omega(\mathbf{x}',0)$, where $\Omega(\mathbf{x}',0)$ is analytic in $\mathbf{x}'$. For simplicity, and because we want the test particle to measure the instantaneous values of A_z and V at the end of this evolution (and not their average values during or after the evolution), we "turn on" H' instantaneously at time T, i.e., we multiply H' by $\delta(t-T)$. The state of the moving charge and the test particle after the postselection is then

$$\Phi(\mathbf{x},T)\Omega(\mathbf{x}',T)=\langle\Psi_{fin}|e^{-i[(\mathbf{p}'-q'\mathbf{A})^2/2m+q'V]/\hbar}\times e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle\Phi(\mathbf{x},0)\Omega(\mathbf{x}',0). \quad (20)$$

The potentials V and $\mathbf{A}$ in Eq. (20) are defined by Eqs. (18) and (19). But we now show that the weak speed $\langle v_z\rangle_w$ replaces v_z in Eqs. (18)–(20). Here we present a short proof, while Appendix B contains a long rigorous proof.

Let us focus on the right-hand side of Eq. (20) and note that we can expand the first exponential,

$$e^{-i[(\mathbf{p}'-q'\mathbf{A})^2/2m+q'V]/\hbar}, \quad (21)$$

as a power series in v_z. Thus, the right-hand side of Eq. (20) is a sum of terms of the form

$$\langle\Psi_{fin}|v_z^n e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle \quad (22)$$

multiplied on either side by functions that do not depend on v_z. But we have, for any n and in the limit $N\to\infty$,

$$\begin{aligned}\langle\Psi_{fin}|v_z^n e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle &= \left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n\langle\Psi_{fin}|e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle\\ &=\langle\Psi_{fin}|\Psi_{in}\rangle\left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n e^{-ip_z\langle v_z\rangle_w T/\hbar}\\ &=\langle\Psi_{fin}|\Psi_{in}\rangle(\langle v_z\rangle_w)^n e^{-ip_z\langle v_z\rangle_w T/\hbar}. \end{aligned}\quad (23)$$

[Compare Eqs. (6)–(11).] So we can simply replace v_z by $\langle v_z\rangle_w$ everywhere it appears in the series. We drop the factor $\langle\Psi_{fin}|\Psi_{in}\rangle$ (to normalize) and obtain

$$\Phi(\mathbf{x},T)\Omega(\mathbf{x}',T)=e^{-i[(\mathbf{p}'-q'\mathbf{A})^2/2m+q'V]/\hbar}\times\Phi(x,y,z-\langle v_z\rangle_w T,0)\Omega(\mathbf{x}',0), \quad (24)$$

where

$$A_z=(\langle v_z\rangle_w/c)V=(\langle v_z\rangle_w/c)V(\mathbf{x}'-\mathbf{x},0;\langle v_z\rangle_w). \quad (25)$$

Since $V(\mathbf{x}'-\mathbf{x},0;\langle v_z\rangle_w)$ equals $V(\mathbf{x}')$ as defined in Eq. (18) with $\langle v_z\rangle_w$ taking the place of v_z, the scalar and vector potentials are exactly the potentials of a charge moving with weak speed $\langle v_z\rangle_w$ (folded with the width of the localized state Φ) and have the corresponding effect on the test particle. Now if $\langle v_z\rangle_w$ exceeds the speed of light, V and A_z correspond to Cherenkov radiation, the shock wave of a charged particle moving faster than light through a medium.

Cherenkov radiation is a striking illustration of the principle that all weak values measured on a preselected and postselected ensemble are consistent. There is more consistency here than what we have noted. We have shown that a particle emits weak Cherenkov radiation consistent with its superluminal weak speed. But we need not limit ourselves to the Hamiltonian H' in Eq. (17). Given any Hamiltonian $H'(v_z)$ that is analytic in v_z, we can write the time evolution operator $\exp[-i\int H'(v_z)dt/\hbar]$ as a power series in v_z, and then, as before, replace v_z by $\langle v_z\rangle_w$. And what holds for weak speed holds for other weak values.

With our effective two-particle interaction H', we have neglected the radiation modes of the electromagnetic field, just as we often neglect these radiation modes in treating the interaction between two charged particles via the Coulomb potential. When can we consistently neglect the radiation modes? A particle of charge q reveals its position through its electromagnetic field; each mode of the electromagnetic field is, in effect, a measuring device. What assures us that the superposition of localized states in Eq. (20) lasts a time T, if each localized state has a distinct electromagnetic field? In other words, how can we postselect the state $|\Psi_{fin}\rangle$ if the radiation modes can reduce the superposition to a localized state corresponding to one eigenvalue of v_z?

The answer to this question depends on the magnitude of the charge q. If q is large, $\Phi(\mathbf{x},t)$ will not remain a superposition of localized states for long. Each state in the superposition corresponds to the charge moving at a different point along the z axis, localized to within $\Delta z\approx\epsilon$. We assume that this uncertainty conforms to Eq. (15). But if q is large enough, the radiation modes will measure the location of the charge and reduce the uncertainty Δz to less than what Eq. (15) allows, thereby reducing the superposition in Eq. (20). Conversely if q is small, vacuum fluctuations will dominate, and the radiation modes will not reduce the uncertainty Δz to less than what Eq. (15) allows.

We can sharpen this question by imagining an observer at a distance D from the moving charge, who may or may not measure its electric field to determine its position (and thus its speed). If there is a measurement, it reduces the superposition in Eq. (20) to a single localized state; then we cannot postselect $|\Psi_{fin}\rangle$ and there will be no Cherenkov radiation. But if there is no measurement, and we postselect $|\Psi_{fin}\rangle$, there *will* be Cherenkov radiation. Can this observer violate causality? As long as $D\leqslant cT$, there is no problem: the observer is close enough to the particle to causally affect the outcome (whether or not it emits Cherenkov radiation). But for $D>cT$, the observer cannot causally affect the particle before it emits Cherenkov radiation. We are left with an apparent violation of causality; how can the radiation from the particle be consistent with later measurements?

To answer the question, let us suppose that the observer locates the particle by measuring its electric field. At a distance D from the particle, the electric-field strength E is $E=q/D^2$, thus $D=\sqrt{q/E}$. Then $\Delta D=(D^3/2q)\Delta E$. Inferring the position z of the particle from this measurement of E, we have $\Delta z\approx(D^3/2q)\Delta E$. The condition for a weak measurement of v_z is Eq. (15), with Δz taking the place of ϵ; that is,

$$\sqrt{N}(D^3/2q)\Delta E\approx\sqrt{N}(\Delta z)\gg\langle v_z\rangle_w T. \quad (26)$$

Since we assume $D\geqslant cT$, Eq. (26) implies $\sqrt{N}D^2\Delta E\gg 2q$. Now vacuum fluctuations in a region of volume D^3, over a time D/c, induce uncertainty in the electric field that is roughly $\Delta E\approx\sqrt{\hbar c}/D^2$ in magnitude [7]. Thus

$$\hbar c>4q^2/N \quad (27)$$

is the condition for weak measurement and Cherenkov radiation. If q satisfies Eq. (27), then weak Cherenkov radiation is consistent with causality. Indeed, even a strong interaction with the electromagnetic field can show Cherenkov radiation: for any given q, N must satisfy Eq. (27), and then measurements will show superluminal weak speed and Cherenkov radiation. For $q \approx e$, N is approximately the inverse fine-structure constant; for larger q, N must be larger, as well.

Thus Cherenkov radiation does not, by itself, imply superluminal weak speed; we must still postselect $|\Psi_{fin}\rangle$. Given the condition $\hbar c > 4q^2/N$, postselection of $|\Psi_{fin}\rangle$ implies Cherenkov radiation, but the reverse does not hold: Cherenkov radiation does not imply postselection of $|\Psi_{fin}\rangle$. Without postselection, Cherenkov radiation may be an error, a fluctuation of the vacuum.

In this example, we preselect $|\Psi_{in}\rangle$ and postselect $|\Psi_{fin}\rangle$ to get superluminal weak speed. In Eq. (5), which defines these states, all the coefficients are real, and therefore the weak speed is real. For other preselections and postselections, however, the weak speed could be complex. Complex weak values can induce nonunitary time evolution. We will present elsewhere an example of an imaginary weak dipole moment which shows a remarkable interplay between imaginary weak values and entanglement. Here, however, we discuss only real weak values.

V. RELATIVISTIC CAUSALITY

Weak measurements—measurements that yield weak values—are internally consistent because they obey two rules. On one hand, they are weak, hence they hardly disturb the measured system. On the other hand, they are inaccurate and can yield, ''by error,'' weak values. These two rules are intimately related. In our example, the change in the initial wave function $\Phi(\mathbf{x},0)$ is proportional to p_z. Thus, for the measurement to be weak, p_z must be bounded. But if p_z is bounded, then the wave function is analytic [3] in z. And since $\Phi(\mathbf{x},0)$ is analytic in z, the probability density does not vanish for any interval in z. Thus we can localize the particle by error in a region it could not have reached without superluminal speed. What if we were to try to eliminate the possibility of error, either by choosing the initial wave function to be a Dirac delta function, or by otherwise imposing a sharp cutoff on the initial wave function? In either case, the initial wave function would not be an analytic function. But then the expansion of Eqs. (7) and (10) in powers of p_z would not be valid. The exponential of $-ip_z v_z t/\hbar$ in Eq. (6) is a unitary operator that translates $\Phi(\mathbf{x},0)$ to $\Phi(x,y,z-v_z t,0)$. This unitary operator acts on any wave function with a Fourier transform. But the Taylor-series expansion of this unitary operator applied to $\Phi(\mathbf{x},0)$,

$$\sum_{m=0}^{\infty} \frac{(-ip_z v_z t/\hbar)^m}{m!}\Phi(\mathbf{x},0), \tag{28}$$

equals the Taylor-series expansion of $\Phi(x,y,z-v_z t,0)$ around $\Phi(\mathbf{x},0)$ only if $\Phi(\mathbf{x},0)$ is an analytic function. Thus the weak value $\langle v_z\rangle_w$ emerges in this experiment only if the initial wave function $\Phi(\mathbf{x},0)$ is analytic.

Once we understand the role of analyticity in the emergence of $\langle v_z\rangle_w$, we can answer another question: How can $\langle v_z\rangle_w > c$ be consistent with relativistic causality? We have seen that the particle moves with velocity $\langle v_z\rangle_w$ only if $\Phi(\mathbf{x},0)$ is analytic. But if $\Phi(\mathbf{x},0)$ is analytic, then its value and the value of its derivatives at any one point determine its value at all points. Hence $\Phi(\mathbf{x},t) = \Phi(x,y,z-\langle v_z\rangle_w t,0)$ does not transmit any message, because it is the same message for all $\mathbf{x}$ and t. Since $\Phi(\mathbf{x},t)$ does not transmit any message, it does not, in particular, transmit a superluminal message, and there is no violation of relativistic causality.

Thus superluminal weak speed is consistent with relativistic causality and with other measurements. There are two distinct ways in which weak measurements can be consistent. On one hand, if a weak measurement of v_z on a preselected and postselected ensemble yields $\langle v_z\rangle_w > c$, any weak measurement of the electromagnetic field *on the same preselected and postselected ensemble* will show Cherenkov radiation. That is, weak measurements are consistent as long as they apply to the same preselected and postselected ensemble. On the other hand, if measurements do not apply to the same preselected and postselected ensemble, they are consistent even if they yield different measured values. For example, we can follow a weak measurement of v_z with either a postselection or a precise measurement of v_z. If we postselect the state $|\Psi_{fin}\rangle$, we interpret the result of the weak measurement as the weak value $\langle v_z\rangle_w$; if we precisely (re)measure v_z, we may interpret the result of the weak measurement as an error. But these two interpretations of a measured value are consistent, for they apply to different ensembles—the former to a preselected and postselected ensemble and the latter to a preselected ensemble. Thus, how we interpret a measured value depends on what we choose to measure next. Here we have considered weak measurements on a single preselected and postselected ensemble. Together, these measurements yield a consistent picture of a charge moving in vacuum at superluminal speed and emitting Cherenkov radiation.

APPENDIX A

We will prove [8] the following representation for $V(\mathbf{x}',t;v_z)$:

$$V(\mathbf{x}',t;v_z) = q\int_{-\infty}^{\infty} d\tau \, \frac{\delta(t-\tau-|\mathbf{x}'-\mathbf{x}|/c)}{|\mathbf{x}'-\mathbf{x}|}. \tag{A1}$$

Here $|\mathbf{x}'-\mathbf{x}| = [(x')^2+(y')^2+(z'-v_z\tau)^2]^{1/2}$. We evaluate the δ function at its zeros according to the rule

$$\delta(g(\tau)) = \sum_i \frac{\delta(\tau-\tau_i)}{|dg(\tau)/d\tau|}, \tag{A2}$$

where τ_i satisfies $g(\tau_i)=0$ and here

$$g(\tau) = t-\tau-[(x')^2+(y')^2+(z'-v_z\tau)^2]^{1/2}/c. \tag{A3}$$

To obtain the zeros, we solve the quadratic equation

$$c^2(t-\tau)^2 = (x')^2+(y')^2+(z'-v_z\tau)^2 \tag{A4}$$

and require $t \geqslant \tau$. There is one zero for $|v_z| < c$,

$$c\tau=\frac{ct-v_z z'/c+\{[(x')^2+(y')^2](1-v_z^2/c^2)+(z'-v_z t)^2\}^{1/2}}{1-v_z^2/c^2}, \tag{A5}$$

and the integral yields

$$V(\mathbf{x}',t;v_z)=q\{[(x')^2+(y')^2](1-v_z^2/c^2)+(z'-v_z t)^2\}^{-1/2}, \tag{A6}$$

as before. This representation of $V(\mathbf{x}',t;v_z)$ will be very useful in Appendix B.

APPENDIX B

We will show that the weak speed $\langle v_z\rangle_w$ replaces v_z in V and $\mathbf{A}$ in Eq. (20). We first show it in the limit $m\to\infty$, i.e., we first consider only the scalar potential V. Then we generalize to finite m and consider $\mathbf{A}$ too.

Let us focus on the term in angular brackets in Eq. (20) and begin by noting that $V(\mathbf{x}')$ as defined in Eq. (18) can also be written as $V(\mathbf{x}'-\mathbf{x},0;v_z)$, as defined in Eq. (16). Hence (in the limit $m\to\infty$) we can write the term in angular brackets as

$$\begin{aligned}&\langle\Psi_{fin}|e^{-iq'V/\hbar}e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle\\&=\langle\Psi_{fin}|e^{-iq'V(\mathbf{x}'-\mathbf{x},0;v_z)/\hbar}e^{-ip_z v_z T/\hbar}|\Psi_{in}\rangle\\&=\langle\Psi_{fin}|e^{-ip_z v_z T/\hbar}e^{-iq'V(\mathbf{x}'-\mathbf{x},T;v_z)/\hbar}|\Psi_{in}\rangle.\end{aligned} \tag{B1}$$

The trick is to take the dependence on v_z out of $V(\mathbf{x}'-\mathbf{x},T;v_z)$ and put it in a more convenient place. To this end, we refer to the representation in Eq. (A1) and note that all the dependence on v_z is contained in the expression $|\mathbf{x}'-\mathbf{x}|$ which, for $V(\mathbf{x}'-\mathbf{x},T;v_z)$, equals $[(x'-x)^2+(y'-y)^2+(z'-z-v_z\tau)^2]^{1/2}$. It follows that the combination

$$M=e^{ip_z' v_z\tau/\hbar}e^{-iq'V(\mathbf{x}'-\mathbf{x},T;v_z)/\hbar}e^{-ip_z' v_z\tau/\hbar} \tag{B2}$$

is actually *independent* of v_z and we can write the term in angular brackets as

$$\langle\Psi_{fin}|e^{-ip_z v_z T/\hbar}e^{-ip_z' v_z\tau/\hbar}Me^{ip_z' v_z\tau/\hbar}|\Psi_{in}\rangle, \tag{B3}$$

where M is independent of v_z. We would like to move M out of the angular brackets. Indeed we can do so, even though M does not commute with $e^{ip_z' v_z\tau/\hbar}$. The reason is that we can always write $\Omega(\mathbf{x}',0)$ as a sum of Fourier components. For each Fourier component in the sum, we can move M out of the angular brackets, and later move it back in; hence we can do so for the sum itself. Thus we can rewrite Eq. (B3) as

$$\begin{aligned}&\langle\Psi_{fin}|e^{-ip_z v_z T/\hbar}e^{-i(p_z'-\overline{p_z'})v_z\tau/\hbar}|\Psi_{in}\rangle M\\&=\langle\Psi_{fin}|\Psi_{in}\rangle e^{-ip_z\langle v_z\rangle_w T/\hbar}e^{-i(p_z'-\overline{p_z'})\langle v_z\rangle_w\tau/\hbar}M,\end{aligned} \tag{B4}$$

where $\overline{p_z'}$ represents an eigenvalue of p_z' for a given Fourier component. [We have taken the limit $N\to\infty$; compare Eqs. (6)–(11).] Now we can pull $e^{i\overline{p_z'}\langle v_z\rangle_w\tau/\hbar}$ back to the right side of M, turn $\overline{p_z'}$ back into p_z', drop the factor $\langle\Psi_{fin}|\Psi_{in}\rangle$ (to normalize), and rewrite the term in angular brackets as

$$\begin{aligned}&e^{-ip_z\langle v_z\rangle_w T/\hbar}e^{-ip_z'\langle v_z\rangle_w\tau/\hbar}Me^{ip_z'\langle v_z\rangle_w\tau/\hbar}\\&=e^{-ip_z\langle v_z\rangle_w T/\hbar}e^{-iq'V(\mathbf{x}'-\mathbf{x},T;\langle v_z\rangle_w)/\hbar}.\end{aligned} \tag{B5}$$

Applying Eq. (B5) to the combined state $\Phi(\mathbf{x},0)\Omega(\mathbf{x}',0)$ of the moving charge and the test particle, we obtain at time T,

$$\begin{aligned}&\Phi(\mathbf{x},T)\Omega(\mathbf{x}',T)\\&=e^{-ip_z\langle v_z\rangle_w T/\hbar}e^{-iq'V(\mathbf{x}'-\mathbf{x},T;\langle v_z\rangle_w)/\hbar}\Phi(\mathbf{x},0)\Omega(\mathbf{x}',0)\\&=e^{-iq'V(\mathbf{x}'-\mathbf{x},0;\langle v_z\rangle_w)/\hbar}\Phi(x,y,z-\langle v_z\rangle_w T,0)\Omega(\mathbf{x}',0).\end{aligned} \tag{B6}$$

Equation (B6) corresponds to Eqs. (24) and (25) in the limit $m\to\infty$.

Now let m be finite. Since $A_z(\mathbf{x}')$ equals $(v_z/c)V(\mathbf{x}')$, we can define a representation of $A_z(\mathbf{x}',t;v_z)$ to be (v_z/c) times the representation of $V(\mathbf{x}',t;v_z)$ in Eq. (A1). But how do we deal with this extra dependence on v_z in $A_z(\mathbf{x}',t;v_z)$? We can expand the exponential term

$$e^{-i[(\mathbf{p}'-q'\mathbf{A})^2/2m+q'V]/\hbar} \tag{B7}$$

in Eq. (20) as a Taylor series. If we then replace $A_z(\mathbf{x}')=(v_z/c)V(\mathbf{x}'-\mathbf{x},0;v_z)$ by its representation, there will be powers of v_z in the series. But we have, for any n and in the limit $N\to\infty$,

$$\begin{aligned}&\langle\Psi_{fin}|v_z^n e^{-i(p_zT+p_z'\tau-\overline{p_z'}\tau)v_z/\hbar}|\Psi_{in}\rangle\\&=\left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n\langle\Psi_{fin}|e^{-i(p_zT+p_z'\tau-\overline{p_z'}\tau)v_z/\hbar}|\Psi_{in}\rangle\\&=\langle\Psi_{fin}|\Psi_{in}\rangle\left(\frac{i\hbar}{T}\frac{\partial}{\partial p_z}\right)^n e^{-i(p_zT+p_z'\tau-\overline{p_z'}\tau)\langle v_z\rangle_w/\hbar}\\&=\langle\Psi_{fin}|\Psi_{in}\rangle(\langle v_z\rangle_w)^n e^{-i(p_zT+p_z'\tau-\overline{p_z'}\tau)\langle v_z\rangle_w/\hbar},\end{aligned} \tag{B8}$$

so we can replace v_z by $\langle v_z\rangle_w$ everywhere it appears in the series. Then we obtain Eqs. (24) and (25) as the generalization of Eq. (B6).

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[2] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).

[3] Y. Aharonov, S. Popescu, D. Rohrlich, and L. Vaidman, Phys. Rev. A **48**, 4084 (1993).

[4] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).

[5] Suppose at time t_0 we measure $v_z t_0$ with precision $ct_0/\sqrt{N}$, e.g., the interaction Hamiltonian for the measurement is $H_{int} = \delta(t-t_0)P_d v_z \sqrt{N}/c$, where P_d is the momentum conjugate to the position Q_d of a pointer on the measuring device. Then since $v_z=(c/N)\Sigma\sigma_z^{(i)}$, the strength of the interaction with each state $(|\uparrow_i\rangle+|\downarrow_i\rangle)/\sqrt{2}$ is proportional to $1/\sqrt{N}$; and the probability that the measurement leaves this state unchanged equals the expectation value $\langle\cos^2(P_d/\sqrt{N})\rangle$. The probability that the measurement leaves the whole state $|\Psi_{in}\rangle$ unchanged is $\langle\cos^{2N}(P_d/\sqrt{N})\rangle$, which, for large N, approaches $\langle e^{-P_d^2}\rangle$ and can be arbitrarily close to 1.

[6] We can obtain a vector potential $\mathbf{A}(\mathbf{x}',t;v_z)$, as we obtained $V(\mathbf{x}',t;v_z)$, via a Lorentz boost of the Coulomb potential $V(\mathbf{x}',t;0)$. Or we can obtain it from a retarded Green function $G(\mathbf{x}',t;\mathbf{x},\tau)$ satisfying the wave equation

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2-(\boldsymbol{\nabla}')^2\right]G_z(\mathbf{x}',t;\mathbf{x},\tau)=4\pi v_z q\,\delta(\mathbf{x}'-\mathbf{x})\,\delta(t-\tau),$$

just as we can obtain $V(\mathbf{x}',t;v_z)$ from a retarded Green function satisfying the equation given in Ref. [8].

[7] See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, London, 1967), p. 35.

[8] This representation of $V(\mathbf{x}',t;v_z)$ (in the Lorentz gauge) comes from a retarded (causal) Green function $G(\mathbf{x}',t;\mathbf{x},\tau)$ satisfying the wave equation

$$\left[\left(\frac{1}{c^2}\frac{\partial}{\partial t}\right)^2-(\boldsymbol{\nabla}')^2\right]G(\mathbf{x}',t;\mathbf{x},\tau)=4\pi q\,\delta(\mathbf{x}'-\mathbf{x})\,\delta(t-\tau).$$

The solution is

$$G(\mathbf{x}',t;\mathbf{x},\tau)=q\,\frac{\delta(t-\tau-|\mathbf{x}'-\mathbf{x}|/c)}{|\mathbf{x}'-\mathbf{x}|}.$$

It is the scalar potential at $\mathbf{x}',t$ due to the charge at $\mathbf{x},\tau$. See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), pp. 223–225. To obtain Eq. (A1) from $G(\mathbf{x}',t;\mathbf{x},\tau)$, we multiply the above equation by $\delta(x)\,\delta(y)\,\delta(z-v_z\tau)$ and integrate with respect to τ.