CHAPTER 4

Symmetric Diagonally-Dominant Matrices and Graphs

Support theory provides effective algorithms for constructing preconditioners for diagonally-dominant matrices and effective ways to analyze these preconditioners. This chapter explores the structure of diagonally-dominant matrices and the relation between graphs and diagonallly-dominant matrices. The next chapter will show how we can use the structure of diagonally-dominant matrices to analyze preconditioners, and the chapter that follows presents algorithms for constructing algorithms.

1. Incidence Factorizations of Diagonally-Dominant Matrices

Definition 1.1. A square matrix $A \in \mathbb{R}^{n \times n}$ is called *diagonally-dominant* if for every $i = 1, 2, \ldots n$ we have

$$A_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}| \ .$$

Symmetric diagonally dominant matrices have symmetric factorizations $A = UU^T$ such that each column of U has at most two nonzeros, and all nonzeros in each column have the same absolute values. We now establish a notation for such columns.

Definition 1.2. Let $1 \leq i, j \leq n$, $i \neq j$. A length-n *positive edge vector*, denoted $\langle i, -j \rangle$, is the vector

$$\langle i, -j \rangle = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{bmatrix} \vdots \\ +1 \\ \vdots \\ -1 \\ \vdots \end{bmatrix} , \quad \langle i, -j \rangle_k = \begin{cases} +1 & k = i \\ -1 & k = j \\ 0 & \text{otherwise.} \end{cases}$$

A *negative edge vector* $\langle i, j \rangle$ is the vector

$$\langle i, j \rangle = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \begin{bmatrix} \vdots \\ +1 \\ \vdots \\ +1 \\ \vdots \end{bmatrix} , \quad \langle i, j \rangle_k = \begin{cases} +1 & k = i \\ +1 & k = j \\ 0 & \text{otherwise.} \end{cases}$$

The reason for the assignment of signs to edge vectors will become apparent later. A *vertex vector* $\langle i \rangle$ is the unit vector

$$\langle i \rangle_k = \begin{cases} +1 & k = i \\ 0 & \text{otherwise.} \end{cases}$$

We now show that a symmetric diagonally dominant matrix can always be expressed as a sum of outer products of edge and vertex vectors, and therefore, as a symmetric product of a matrix whose columns are edge and vertex vectors.

Lemma 1.3. *Let $A \in \mathbb{R}^{n \times n}$ be a diagonally dominant symmetric matrix. We can decompose A as follows*

$$\begin{aligned}
A &= \sum_{\substack{i<j \\ A_{ij}>0}} |A_{ij}| \langle i, j \rangle \langle i, j \rangle^T \\
&\quad + \sum_{\substack{i<j \\ A_{ij}<0}} |A_{ij}| \langle i, -j \rangle \langle i, -j \rangle^T \\
&\quad + \sum_{i=1}^{n} \left(A_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}| \right) \langle i \rangle \langle i \rangle^T \\
&= \sum_{\substack{i<j \\ A_{ij}>0}} \left(\sqrt{|A_{ij}|}\ \langle i, j \rangle \right) \left(\sqrt{|A_{ij}|}\ \langle i, j \rangle \right)^T \\
&\quad + \sum_{\substack{i<j \\ A_{ij}<0}} \left(\sqrt{|A_{ij}|}\ \langle i, -j \rangle \right) \left(\sqrt{|A_{ij}|}\ \langle i, -j \rangle \right)^T \\
&\quad + \sum_{i=1}^{n} \left(\sqrt{A_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}|}\ \langle i \rangle \right) \left(\sqrt{A_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |A_{ij}|}\ \langle i \rangle \right)^T .
\end{aligned}$$

Proof. The terms in the summations in lines 1 and 4 are clearly equal (we only distributed the scalars), and so are the terms in lines 2 and 5 and in lines 3 and 6. Therefore, the second equality holds.

We now show that the first equality holds. Consider the rank-1 matrix

$$\langle i, -j \rangle \langle i, -j \rangle^T = \begin{bmatrix} \ddots & & & & \\ & +1 & & -1 & \\ & & \ddots & & \\ & -1 & & +1 & \\ & & & & \ddots \end{bmatrix},$$

and similarly,

$$\langle i,j\rangle \langle i,j\rangle^T = \begin{bmatrix} \ddots & & & & \\ & +1 & & +1 & \\ & & \ddots & & \\ & +1 & & +1 & \\ & & & & \ddots \end{bmatrix}$$

(in both matrices the four nonzeros are in rows and columns i and j). Suppose $A_{ij} > 0$ for some $i \neq j$. The only contribution to element i,j from the three sums is from a single term in the first sum, either the term $|A_{ij}| \langle i,j\rangle \langle i,j\rangle^T$ or the term $|A_{ji}| \langle j,i\rangle \langle j,i\rangle^T$, depending on whether $i > j$. The i,j element of this rank-1 matrix is $|A_{ij}| = A_{ij}$. If $A_{ij} < 0$, a similar argument shows that only $|A_{ij}| \langle i,-j\rangle \langle i,-j\rangle^T$ (assuming $i < j$) contributes to the i,j element, and that the value of the element $-|A_{ij}| = A_{ij}$. The third summation ensures that the values of diagonal elements is also correct. $\square$

The matrix decompositions of this form play a prominent role in support theory, so we give them a name:

Definition 1.4. A matrix whose columns are scaled edge and vertex vectors (that is, vectors of the forms $c\langle i,-j\rangle$, $c\langle i,j\rangle$, and $c\langle i\rangle$) is called an *incidence matrix*. A factorization $A = UU^T$ where U is an incidence matrix is called an *incidence factorization*. An incidence factorization with no zero columns, with at most one vertex vector for each index i, with at most one edge vector for each index pair i,j, and whose positive edge vectors are all of the form $c\langle \min(i,j), -\max(i,j)\rangle$ is called a *canonical incidence factorization*.

Lemma 1.5. *Let $A \in \mathbb{R}^{n\times n}$ be a diagonally dominant symmetric matrix. Then A has an incidence factorization $A = UU^T$, and a unique canonical incidence factorization.*

Proof. The existence of the factorization follows directly from Lemma 1.3. We now show that the canonical incidence factorization is uniquely determined by A. Suppose that $A_{ij} = 0$. Then U cannot have a column which is a nonzero multiple of $\langle i,j\rangle$, $\langle i,-j\rangle$, or $\langle -i,j\rangle$, since if it did, there would be only one such column, which would imply $\left(UU^T\right)_{ij} \neq 0 = A_{ij}$. Now suppost that $A_{ij} > 0$. Then U must have a column $\sqrt{A_{ij}}\,\langle i,j\rangle$. Similarly, if $A_{ij} < 0$, then U must have a column $\sqrt{-A_{ij}}\,\langle \min(i,j), -\max(i,j)\rangle$. The uniqueness of the edge vectors implies that the scaled vertex vectors in U are also unique. $\square$

2. Graphs and Their Laplacians Matrices

We now define the connection between undirected graphs and diagonally-dominant symmetric matrices.

Definition 2.1. Let $G = (\{1,2,\ldots n\}, E)$ be an undirected graph on the vertex set $\{1,2,\ldots,n\}$ with no self loops or parallel edges. That is, the edge-set E consists of unordered pairs of unequal integers (i,j) such that $1 \leq i,j \leq n$ and $i \neq j$. The degree of a vertex i is the numbers of edges incident on it. The *Laplacian* of G is the matrix $A \in \mathbb{R}^{n\times n}$ such that

$$A_{ij} = \begin{cases} \text{degree}(i) & i = j \\ -1 & (i,j) \in E \text{ (the index pair is unordered)} \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.2. *The Laplacian of an undirected graph is symmetric and diagonally dominant.*

PROOF. The Laplacian is symmetric because the graph is undirected. The Laplacian is diagonally dominant because the number of off-diagonal nonzero entries in row i is exactly degree(i) and the value of each such nonzero is -1. $\square$

We now generalize the definition of Laplacians to undirected graphs with positive edge weights.

DEFINITION 2.3. Let $G = (\{1, 2, \ldots n\}, E, c)$ be a weighted undirected graph on the vertex set $\{1, 2, \ldots, n\}$ with no self loops, and with a weight function $c : E \to \mathbb{R} \setminus \{0\}$. The *weighted Laplacian* of G is the matrix $A \in \mathbb{R}^{n\times n}$ such that

$$A_{ij} = \begin{cases} \sum_{(i,k)\in E} |c(i,k)| & i = j \\ -c(i,j) & (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

If some of the edge weights are negative, we call G a *signed* graph, otherwise, we simply call it a weighted graph.

LEMMA 2.4. *The Laplacian of a weighted undirected graph (signed or unsigned) is symmetric and diagonally dominant.*

PROOF. Again symmetry follows from the fact that the graph is undirected. Diagonal dominance follows from the following equation, which holds for any i, $1 \le i \le n$:

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}| = \sum_{\substack{j=1 \\ j\neq i \\ A_{ij}\neq 0}}^{n} |A_{ij}| = \sum_{(i,j)\in E}^{n} |A_{ij}| = \sum_{(i,j)\in E}^{n} |c(i,j)| = A_{ii} \ .$$

$\square$

If we allow graphs to have nonnegative vertex weights as well as edge weights, then this class of graphs becomes completely isomorphic to symmetric diagonally-dominant matrices.

DEFINITION 2.5. Let $G = (\{1, 2, \ldots n\}, E, c, d)$ be a weighted undirected graph on the vertex set $\{1, 2, \ldots, n\}$ with no self loops, and with weight functions $c : E \to \mathbb{R} \setminus \{0\}$ and $d : \{1, \ldots, n\} \to \mathbb{R}_+ \cup \{0\}$. The *Laplacian* of G is the matrix $A \in \mathbb{R}^{n\times n}$ such that

$$A_{ij} = \begin{cases} d(i) + \sum_{(i,k)\in E} |c(i,k)| & i = j \\ -c(i,j) & (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

A vertex i such that $d(i) > 0$ is called a *strictly dominant* vertex.

LEMMA 2.6. *The Laplacians of the graphs defined in Definition 2.5 are symmetric and diagonally dominant. Furthermore, these graphs are isomorphic to symmetric diagonally-dominant matrices under this Laplacian mapping.*

PROOF. Symmetry again follows from the fact that the graph is undirected. For any i we have

$$\sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}| = \sum_{\substack{j=1\\ j\neq i\\ A_{ij}\neq 0}}^{n} |A_{ij}| = \sum_{(i,j)\in E}^{n} |A_{ij}| = \sum_{(i,j)\in E}^{n} |c(i,j)| = A_{ii} - d(i) \ ,$$

so

$$A_{ii} = d(i) + \sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}| \geq \sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}|$$

because $d(i) \geq 0$. This shows that Laplacians are diagonally dominant. The isomorphism follows from the fact that the following expressions uniquely determine the graph $(\{1,\ldots,n\}, E, c, d)$ associated with a diagonally-dominant symmetric matrix:

$$\begin{array}{rcl} (i,j)\in E & \text{iff} & i\neq j \text{ and } A_{ij}\neq 0 \\ c(i,j) & = & -A_{ij} \\ d(i) & = & A_{ii} - \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} |A_{ij}| \ . \end{array}$$

□

We prefer to work with vertex weights rather than allowing self loops because edge and vertex vectors are algebraically different. As we shall see below, linear combinations of edge vectors can produce new edge vectors, but never vertex vectors. Therefore, it is convenient to distinguish edge weights that correspond to scaling of edge vectors from vertex weights that correspond to scaling of vertex vectors.

In algorithms, given an explicit representation of a diagonally-dominant matrix A, we can easily compute an explicit representation of an incidence factor U (including the canonical incidence factor if desired). Sparse matrices are often represented by a data structure that stores a compressed array of nonzero entries for each row or each column of the matrix. Each entry in a row (column) array stores the column index (row index) of the nonzero, and the value of the nonzero. From such a representation of A we can easily construct a sparse representation of U by columns. We traverse each row of A, creating a column of U for each nonzero in the upper (or lower) part of A. During the traversal, we can also compute all the $d(i)$'s. The conversion works even if only the upper or lower part of A is represented explicitly.

We can use the explicit representation of A as an implicit representation of U, with each off-diagonal nonzero of A representing an edge-vector column of U. If A has no strictly-dominant rows, that is all. If A has strictly dominant rows, we need to compute their weights using a linear traversal of A.

3. Laplacians and Resistive Networks

Weighted (but unsigned) Laplacians model voltage-current relationships in electrical circuits made up of resistors. This fact has been used in developing some support preconditioners. This section explains how Laplacians model resistive networks.

Let $G_A = (\{1, \dots, n\}, E, c)$ be an edge-weighted graph and let A be its Laplacian. We view G_A as an electrical circuit with n nodes (connection points) and with $m = |E|$ resistors. If $(i, j) \in E$, then there is a resistor with capacitance $c(i, j)$ between nodes i and j (equivalently, with resistence $1/c(i, j)$).

Given a vector x of node potentials, the voltage drop across resistor (i, j) is $|x_i - x_j|$. If the voltage drop is nonzero, current will flow across the resistor. Current flow has a direction, so we need to orient resistors in order to assign a direction to current flows. We arbitrarily orient resistors from the lower-numbered node to the higher-numbered one. In that direction, the directional voltage drop across (i, j) is

$$x_{\min(i,j)} - x_{\max(i,j)} = \langle \min(i, j), -\max(i, j)\rangle^T x \, .$$

We denote $E = \{(i_1, j_1), (i_2, j_2), \dots, (i_m, j_m)\}$ and we denote

$$\tilde{U} = \left[\ \langle \min(i_1, j_1), -\max(i_1, j_1)\rangle \quad \cdots \quad \langle \min(i_m, j_m), -\max(i_m, j_m)\rangle \ \right] \, .$$

Using this notation, the vector $\tilde{U}^T x$ is the vector of all directional voltage drops. The directional current flow across resistor (i, j) is $c(i, j)(x_{\min(i,j)} - x_{\max(i,j)})$. We denote by C the diagonal matrix with $C_{k,k} = c(i_k, j_k)$. Then the vector $C\tilde{U}^T x$ is the vector of directional current flow across the m resistors.

Next, we compute nodal currents, the net current flowing in or out of each node. If we set the potential of node i to x_i, this means that we have connected node i to a voltage source, such as a battery. The voltage source keeps the potential at i at exactly x_i volts. To do so, it might need to send current into x_i or to absorb currect flowing out of x_i. These current flows also have a direction: we can either compute the current flowing into i, or the current flowing out of i. We will arbitrarily compute the current flowing into i. How much current flows into node i? Exactly the net current flowing into it from its neighbors in the circuit, minus the current flowing from it to its neighbors. Let $(i_k, j_k) \in E$. If $i_k < j_k$, then $\left(C\tilde{U}^T x\right)_k$ represents the current $c(i_k, j_k)(x_{\min(i_k,j_k)} - x_{\max(i_k,j_k)})$ flowing from i_k to j_k, which is positive if $x_{i_k} > x_{j_k}$. If $i_k > j_k$ then current flowing from i_k to j_k will have a negative sign (even though it too is flowing out of $i_k$if $x_{i_k} > x_{j_k}$) and we have to negate it before we add it to the total current flowing from i_k to its neighbors, which is exactly the net current flowing into i_k from the voltage source. That is, the total current from a node i to its neighbors is

$$\sum_{\substack{(i_k,j_k)\in E \\ i_k = i}} (-1)^{i > j_k} \left(C\tilde{U}^T x\right)_k = \tilde{U}_{i,:} \, C\tilde{U}^T x \, .$$

Therefore, the vector of net node currents is exactly $\tilde{U}C\tilde{U}^T = A$. We can also compute the total power dissipated by the circuit. We multiply the current flowing across each resistor by the voltage drop and sum over resistors, to obtain $\left(x^T\tilde{U}\right)\left(C\tilde{U}^T x\right) = x^T\tilde{U}C\tilde{U}^T x$.

We summarize the various voltage-current relationships:

x	nodal voltages
$\tilde{U}^T x$	voltage drops across resistors
$C\tilde{U}^T x$	current flows across resistors
$\tilde{U}C\tilde{U}^T x = Ax$	net nodal currents
$x^T\tilde{U}C\tilde{U}^T x = x^T Ax$	total power dissipated by the circuit

These relationships provide physical interpretations to many quantities and equations that we have already seen. Given a vector x of nodal voltages, Ax is the corresponding vector of nodal currents, where A is the Laplacian of the circuit. Conversely, given a vector b of nodal currents, solving $Ax = b$ determines the corresponding nodal voltages. The maximal eigenvalue of A measures the maximum power that the circuit can dissipate under a unit vector of voltages, and the minimal nonzero eigenvalue measures the minimum power that is dissipated under a unit voltage vector that is orthogonal to the constant vector (which spans the null space of A).

Laplacians with strictly-dominant rows arise when the boundary conditions, the known quantities in the circuit, include a mixture of nodal voltages and nodal currents. Suppose that we know the currents in all the nodes except for one, say n, where we know the voltage, not the current. Let us assume that the voltage at node n is 5 Volts. We want to compute the voltages at nodes 1 to $n-1$. We cannot use the linear system $\tilde{U}C\tilde{U}^T x = Ax = b$ directly, because we do not know b_n, and on the other hand, we do know that $x_n = 5$. Therefore, we drop the last row from this linear system, since we do not know its right-hand side. We have

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n-1} & A_{1,n} \\ \vdots & & \vdots & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,n-1} & A_{n-1,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 5 \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

or

$$\begin{bmatrix} A_{1,1} & \cdots & A_{1,n-1} \\ \vdots & & \vdots \\ A_{n-1,1} & \cdots & A_{n-1,n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} - 5 \begin{bmatrix} A_{1,n-1} \\ \vdots \\ A_{n-1,n-1} \end{bmatrix} .$$

We have obtained a new square and symmetric coefficient matrix and a known right-hand side. The new matrix is still diagonally dominant, but now has strictly-dominant rows: if $A_{k,n} \neq 0$, then row k in the new coefficient matrix is now strictly-dominant, since we removed $A_{k,n}$ from it. If we know the voltages at other nodes, we repeat the process and drop more rows and columns, making the remaining coefficient matrix even more dominant.

Given two resistive networks with the same number of unknown-voltage nodes, what is the interpretation of a path embedding π? An embedding of G_A in G_B shows, for each edge (i,j) in G_A, a path between i and j in G_B. That path can carry current and its conductance, which is the inverse of the sum of resistances along the path, serves as a lower bound on the conductance between i and j in G_B. Intuitively, an embedding allows us to show that G_B is, up to a certain factor, “as good as” G_A, in the sense that for a given vector of voltages, currents in G_B are not that much smaller than the currents in G_A.