

Statistical Learning, Fall 2024-25

Homework exercise 2

Due date: 28 Nov. 2024 before class

1. Population Optimizer of absolute loss

Prove that for absolute loss: $L_{\text{abs}}(Y, f(X)) = |Y - f(X)|$, EPE is minimized by setting $f^*(x) = \text{Median}(Y|X = x)$

Hint: you may find the following identity useful:

$$\int_{y>c} (y - c) dP(y) = \int_{y>c} Pr(Y > y) dy$$

(a) **Generalization to quantile loss** The τ th quantile loss for $0 < \tau < 1$ is defined as:

$$L_{\tau}(Y, f(X)) = \begin{cases} \tau \times (Y - f(X)) & \text{if } Y - f(X) > 0 \\ -(1 - \tau) \times (Y - f(X)) & \text{otherwise} \end{cases}$$

Prove that the EPE is minimized by setting $f^*(x)$ to be the τ th quantile of $P(Y|X = x)$, i.e., $P(Y \leq f^*(x)|X = x) = \tau$

2. **ESL 2.3:** Derive equation (2.24) (expected median distance to origin's nearest neighbor in an ℓ_p ball):

$$d(p, n) = \left(1 - \frac{1}{2}\right)^{1/p}$$

Suggested approach:

- (a) Find the probability that all observations are outside a ball of radius $r < 1$, as a function of r .
- (b) You are looking for r such that this probability is $1/2$.

Plot $d(p, n)$ against p for $n \in \{100, 5000, 100000\}$ and $p \in \{3, 5, 10, 20, 50, 100\}$ (make one curve for every value of n — use the R functions `plot()` and `lines()`) and interpret the graph.

3. **ESL 2.7:** Compare classification performance of k-NN and linear regression on the `zipcode`¹ data, on the task of separating the digits 2 and 3. Use $k \in \{1, 3, 5, 7, 15\}$. Plot training and test error for k-NN choices and linear regression. Comment on the shape of the graph.

4. **ESL 2.9 (second edition only)** Consider a linear regression model, fit by least squares to a set of training examples $T = \{(X_1, Y_1), \dots, (X_N, Y_N)\}$, drawn i.i.d from some population. Let $\hat{\beta}$ be the least

¹Training: <https://hastie.su.domains/ElemStatLearn/datasets/zip.train.gz>
Testing: <https://hastie.su.domains/ElemStatLearn/datasets/zip.test.gz>
Info: <https://hastie.su.domains/ElemStatLearn/datasets/zip.info.txt>

squares estimate. Suppose we also have some other (“test”) data drawn independently from the same distribution $\{(\tilde{X}_1, \tilde{Y}_1), \dots, (\tilde{X}_M, \tilde{Y}_M)\}$. Prove that:

$$\frac{1}{N} \mathbb{E} \left(\sum_{i=1}^N (Y_i - X_i^T \hat{\beta})^2 \right) \leq \frac{1}{M} \mathbb{E} \left(\sum_{i=1}^M (\tilde{Y}_i - \tilde{X}_i^T \hat{\beta})^2 \right),$$

that is, the expected squared error in-sample is always bigger than out of sample in least squares fitting. Note that the values \tilde{X} are also random variables here, and the expectation is over everything that is random, including X, Y and $\hat{\beta}$.

Hint: There are several ways to prove this. One starts from considering the best possible linear model we derived in class:

$$\beta^* = (E(XX^T))^{-1} E(XY),$$

and comparing both sides to it.

Note: Students who find more than one valid way to prove the result will get a bonus grade.

* **Extra credit problem: Optimality of k-NN in fixed dimension**

Assume $X \sim \text{Unif}([0, 1]^p)$, and $Y = f(X) + \epsilon$ with $\epsilon \sim (0, \sigma^2)$ (that is, $f(x) = E(Y|X = x)$).

Assume f is Lipschitz: $\|x_1 - x_2\| < \delta \Rightarrow |f(x_1) - f(x_2)| < c\delta, \forall x_1, x_2 \in [0, 1]^p$. Choose any sequence $k(n)$ such that:

$$\begin{aligned} k(n) &\xrightarrow{n \rightarrow \infty} \infty \\ k(n)/n &\xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Then:

$$\text{EPE}(\text{k-NN using } k(n)) \xrightarrow{n \rightarrow \infty} \text{EPE}(f) = \sigma^2$$

(The proof does not have to be completely formal, for example you can replace a binomial with its normal approximation without proof of the relevant asymptotics).