Oscillator phase states, thermal equilibrium and group representations*

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Eigenstates of the annihilation type operator $U = C + iS$, where C and S are the "cosine" and "sine" operators for harmonic oscillator phase, are shown to be closely related to thermal equilibrium states of the oscillator and to provide a new interpretation of the thermal equilibrium density operator. The problem of creating such states is considered and a general theorem is established leading to the construction of interaction Hamiltonians which transform the eigenstates of U among themselves and, in particular, create them from the oscillator ground state. These Hamiltonians lead to representations of the Lie algebras of $O(2,1)$ and $O(3)$. It is suggested that the mathematical technique used, in which generalized U-type operators provide the link between a group and its representations, has its own intrinsic interest for the study of Lie groups.

1. INTRODUCTION

In studying the quantum theory of harmonic oscillator phase, new Hermitian operators C and S were introduced[1-7] whose spectra coincide with the range of values of the trigonometric functions $\cos\phi$ and $\sin\phi$. Since these operators do not commute with one another, one cannot prepare a state in which the phase is arbitrarily sharply defined except in certain limiting cases. However, one might expect that the operator $U = C + iS$, which is the quantum analog of the quantity $e^{i\phi} = \cos\phi + i\sin\phi$, would define states of maximal phase resolution in some reasonable sense.

The eigenstates of U, referred to here as the phase states, have been studied,[7] and not only provide a physically reasonable description of phase, but also possess other interesting physical properties. These properties stem from the close relationship between phase states and the description of an oscillator in thermal equilibrium with its surroundings, a relationship in which the classical concept of oscillator phase plays an important role.

As an example, consider an oscillator of natural frequency ω in thermal equilibrium at a temperature T. Then the statistical average of the oscillator energy can be shown to be equal to a pure quantum expectation value in a suitable phase state. This implies that measurement of the oscillator energy cannot distinguish a phase state from a thermal equilibrium mixture. The expectation value of any other oscillator observable is obtained by uniform averaging of its quantum expectation value in such a state over a single parameter which, in the limit $kT \gg \hbar\omega$, is identifiable as the classical phase of the oscillator.

Therefore, although the density operator formalism makes it clear that thermal equilibrium cannot be described by a pure quantum state, the phase states provide as close a description as one might hope for within the pure state framework. The additional randomness associated with thermal equilibrium is represented by a uniform distribution over the phase parameter associated with the state.

In view of the foregoing remarks, it becomes a matter of considerable interest to examine the possibility of finding a physical model for the creation of a phase state. In this model the oscillator would be part of a well-defined larger dynamical system, the effect of which would be to subject it to an interaction which would take it from its ground state, for example, to a phase state. Such a model would conceivably have the interesting property of exhibiting, within the framework of pure quantum dynamics, a process very closely related to the approach to thermal equilibrium.

An obvious first step in the search for this model is to find interaction Hamiltonians which generate phase states. The formulation and solution of this problem form the bulk of the present paper. In developing the mathematical techniques for this purpose, we find that the desired Hamiltonians form a representation of a Lie algebra. The elements of the group generated by this algebra are identified with transformations of the spectrum of the operator U, which in turn acts in the underlying space of the representation. We believe that this linking of the group with its representation via an operator is of sufficient generality to be of intrinsic interest for the study of group representations.

This paper is therefore one of largely mathematical nature, motivated and guided by physical considerations. The emergence of a possible new approach to the study of Lie groups was quite unforseen at the outset. We therefore do not include here attempts to extend our approach beyond the groups $O(2, 1)$ and $O(3)$ which arise naturally in our treatment of harmonic oscillator dynamics. These further efforts will be the subject of a future paper.

In Sec. 2, the relation indicated above between the statistics of an oscillator in thermal equilibrium and the phase states is established. A brief discussion of the properties of these states and of the operators defining them is included here and in the following section. Further clarification of their significance as phase states is provided by examining them in the classical limit and showing that they have just the interpretation one would expect in terms of an ensemble in phase space.

Section 3 is concerned with the formulation and solution of the problem of finding the Hamiltonians that generate phase states. The formulation is achieved by establishing a general theorem which characterizes those Hamiltonians which transform the eigenstates of an operator of the type U into themselves. This leads to a whole class of operators $A(k)$ parametrized by a real variable k and including the operator U itself. For each value of k, there is a set of allowable Hamiltonians $H(k)$. It follows from the general conditions of our theorem that the $H(k)$ constitute a representation of a Lie algebra, which turns out to be essentially that of $O(2, 1)$. We show also that the familiar destruction operator, whose eigenstates are the coherent states, is a member of our class in the limit $k \to \infty$. We thus refer to the operators $A(k)$ as generalized destruction operators.

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In Sec. 4 we explicitly exhibit the general form of the unitary transformations which the $H(k)$ induce on the $A(k)$ eigenstates. It is seen that the eigenvalues of $A(k)$ undergo linear fractional transformations. For a given k, these transformations can be classified into two types. In one, the eigenvalues vary periodically in time; in the other they display the interesting property of approaching a given limit as $t \to \infty$ independently of their initial values.

In the last two sections, we take up the group theoretic aspects of our work. In Sec. 5, we show that an extension of the range of the parameter k allows us to generalize the methods of Sec. 3 to include $O(3)$ as well as $O(2, 1)$. By a slight broadening of the concept of eigenstate, we are able to retain the useful transformation properties of the eigenstates of the generalized destruction operator even in the finite dimensional subspaces which support representations of $O(3)$. In the final section, after identifying the relevant representations of $O(3)$ and $O(2, 1)$, we give some characterization of the phase states from the point of view of the theory of group representations. The section closes with a discussion of the derivation of the explicit forms of the matrix representation of the group by means of our formalism.

2. PHASE STATES AND THERMAL EQUILIBRIUM

Consider an oscillator which has been brought into thermal contact with a heat bath of temperature $T = (k\beta)^{-1}$ and allowed to reach equilibrium. If the contact is then broken and the oscillator allowed to evolve as an isolated system, its state is represented by the thermal density operator

$$\rho = (1 - e^{-\beta\hbar\omega}) \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} |n\rangle\langle n| , \tag{2.1}$$

where the $|n\rangle$ are orthonormal eigenstates of the number operator $N = a^+a$.

Formally, we may exhibit a pure quantum state which, for observables diagonal on the oscillator number basis, gives the same expectation values as are obtained from the thermal density operator. This state is

$$|\psi_\beta\rangle = \sqrt{1 - e^{-\beta\hbar\omega}} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega/2} |n\rangle . \tag{2.2}$$

As an example, the familiar result for the average value of the number operator,

$$\langle N\rangle = 1/(e^{\beta\hbar\omega} - 1) \tag{2.3}$$

is obtained as a pure quantum expectation value. For general observables, we may make use of the time dependent state associated with (2.2),

$$|\psi_\beta(t)\rangle = \sqrt{1 - e^{-\beta\hbar\omega}} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega/2 - in\omega t} |n\rangle , \tag{2.4}$$

to obtain ρ by time averaging over the oscillator period τ:

$$\rho = (1/\tau) \int_0^\tau dt\, |\psi_\beta(t)\rangle\langle\psi_\beta(t)| . \tag{2.5}$$

These somewhat formal constructs acquire a more tangible significance from the fact that, for the classical oscillator, averaging over a period is equivalent to averaging over a completely random phase. With this in mind, we rewrite Eq. (2.5) as

$$\rho = \int_0^{2\pi} \frac{d\phi}{2\pi} |\psi_\beta(\phi)\rangle\langle\psi_\beta(\phi)| , \tag{2.6}$$

with $|\psi_\beta(\phi)\rangle$ defined by

$$|\psi_\beta(\phi)\rangle = \sqrt{1 - e^{-\beta\hbar\omega}} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega/2} e^{in\phi} |n\rangle . \tag{2.7}$$

Thus, the time averaging of Eq. (2.5) is replaced by an averaging over a set of time independent states parametrized by the quantity ϕ.

Of course, the resemblance of Eq. (2.6) to classical phase averaging does not guarantee that ϕ is related to the phase. The relation can be established, however, on the basis of a previously published mathematical analysis[7] of oscillator phase operators in which new states of the oscillator, the phase states, were introduced. These are eigenstates of the operator $U = C + iS$, where C and S are the "cosine" and "sine" operators and satisfy

$$[C,N] = iS, \quad [S,N] = -iC, \tag{2.8}$$

or

$$[U,N] = U. \tag{2.9}$$

The simplest choice for U, namely, $U = E$, the unit shift operator, defined by

$$E|n\rangle = (1 - \delta_{o n})\,|n - 1\rangle, \tag{2.10}$$

leads to phase states of precisely the form (2.7) with the eigenvalue z identified as

$$z = |z|\, e^{i\phi} = e^{i\phi - \beta\hbar\omega/2} = \sqrt{\frac{\langle N\rangle}{\langle N\rangle + 1}}\, e^{i\phi} . \tag{2.11}$$

Thus the averaging in Eq. (2.6) is carried out with respect to a parameter which, in the limit $\langle N\rangle \to \infty$ (i.e., $\beta\hbar\omega \ll 1$, or $|z| \to 1$), is formally identifiable with the classical oscillator phase.[8]

This formal identification of ϕ with the classical phase can be made intuitive by noting the following behavior, which is established mathematically in the Appendix: for $\phi = 0$, the expectation value of the coordinate $\langle q\rangle$ becomes indefinitely large with large $\langle N\rangle$. Furthermore, the fractional uncertainty $\delta q/\langle q\rangle$ approaches a nonzero constant value less than unity. Therefore, δq also becomes very large, but remains less than $\langle q\rangle$, indicating that the probability distribution covers primarily the positive real axis.[9] At the same time, the expectation value of the momentum $\langle p\rangle$ vanishes, and the uncertainty δp becomes vanishingly small as $\langle N\rangle$ becomes infinite. Thus p is sharply defined about zero. When $\phi = \pi/2$, the roles of p and q are reversed, with the p distribution being smeared out over positive values and q being sharply defined about the origin.

More descriptively, we can picture a classical ensemble of oscillators with different amplitudes but identical phases. Thus, $\phi = 0$ sees them strung out to the right of the origin and at rest, while $\phi = \pi/2$ sees them all located at the origin with different momenta, but moving in the same direction.

The phase states differ in this respect from the well-known coherent states, which are eigenstates of the annihilation operator with eigenvalue α, expressible as

$$\alpha = (1/\sqrt{2})(\langle q\rangle + i\langle p\rangle) = \sqrt{\langle N\rangle}\, e^{i\phi} . \tag{2.12}$$

Here the parameter ϕ also becomes interpretable[3] as the phase of the oscillator in the limit of large $\langle N\rangle$. But in this case, *all* the dynamical quantities, q, p, N, etc.,

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become essentially classical, having fractional uncertainties which vanish in this limit. The classical behavior is evidenced by the fact, shown by Glauber,[10] that the thermal density operator (2.6) can be expressed in terms of the coherent states as[11]

$$\rho = \frac{1}{\pi\mathfrak{N}} \int d^2\alpha\, e^{-|\alpha|^2/\mathfrak{N}} \,|\, \alpha\rangle\langle\alpha \,|\,. \tag{2.13}$$

The distribution function in Eq. (2.13) is Gaussian, as one would expect classically. In the limit of small β, it goes over into the Boltzman function and becomes interpretable as a probability distribution.[10]

In analogous fashion, the uniform distribution over ϕ in Eq. (2.6) becomes interpretable as a uniform probability distribution in phase for large $\langle N\rangle$. The distinguishing property of the phase states is that, for them, only the phase becomes classically definable, the other quantities retaining finite fractional uncertainties. Thus the density operator acquires a new interpretation. The statistics associated with phase independent variables, such as the energy, are purely quantum-mechanical, while the total statistical picture emerges as a result of uniform phase averaging.

In view of this close relationship between phase states and thermal averaging, it becomes a matter of considerable interest to find a model in which these states are generated as a result of some interaction. It is with this aim in mind that we undertake in this paper a general mathematical formulation of the problem of finding Hamiltonians which generate states of this type from the ground state and transform them into one another.

3. GENERAL FORMULATION

We begin by noting[12] that the phase states form a nondegenerate, overcomplete set of eigenstates of the nonunitary shift operator E, whose spectrum consists of the unit circle in the complex plane. An interior point z of the circle corresponds to a phase state

$$|z\rangle = \sqrt{1-|z|^2} \sum_{n=0}^{\infty} z^n |n\rangle\,. \tag{3.1}$$

These properties bear a strong resemblance to those of the coherent states, where the relevant shift operator is the familiar annihilation operator $E\sqrt{N}$, and the spectrum consists of the entire complex plane.

An important feature of the coherent states is that they retain their character as eigenstates of the annihilation operator under the influence of linear, c-number driving forces, that is, the class of Hamiltonians of the form

$$H = c_1 N + c_2 q + c_3 p \tag{3.2}$$

generates unitary transformations of these states into themselves.

In seeking analogous Hamiltonians for the phases states we were led to the following general formulation of the problem:

(i) Consider an annihilation type operator $A = EF(N)$, so that

$$A\,|n\rangle = F(n)\,|n-1\rangle \tag{3.3}$$

with $F(0) = 0$, but $F(n)$ nonvanishing for $n \neq 0$. There is no loss of generality in assuming $F(n)$ to be real and positive.[7] We assume further than $F(n)$ converges to a finite nonzero limit as $n \to \infty$. A solution of the eigenvalue equation

$$A\,\|\,\alpha\rangle = \alpha\,\|\,\alpha\rangle \tag{3.4}$$

is given by

$$\|\,\alpha\rangle = |\,0\rangle + \sum_{n=1}^{\infty} \frac{\alpha^n}{\prod_{m=1}^{n} F(m)}\,|\,n\rangle, \tag{3.5}$$

in which we assume that the $F(n)$ are such that $\|\,\alpha\rangle$ is normalizable for $|\alpha| < R$ where R is some nonzero positive constant. The double bar notation is used to denote states with the normalization (3.5) in which the coefficient of the ground state is unity and the coefficients of the other states are analytic in α. The eigenstate with unit norm is denoted by $|\,\alpha\rangle$. The spectrum[7] of A then consists of the interior and circumference of a circle of radius R in the complex plane. The states $\|\,\alpha\rangle$ form a nondegenerate, overcomplete[13] set.

(ii) We now show that a necessary and sufficient condition that a Hamiltonian H generates transformations of the $\|\,\alpha\rangle$ into themselves is that[14]

$$[A,H] = f(A), \tag{3.6}$$

where the notation $f(A)$ is understood to mean that the eigenstates of A are also eigenstates of $[A,H]$. It will be seen, in fact, that f is an analytic function of its argument.

The proof of necessity proceeds from the fact that if H generates unitary transformations of the $\|\,\alpha\rangle$ into themselves, it follows that

$$Ae^{-itH}\,|\,\alpha\rangle = \lambda e^{-itH}\,|\,\alpha\rangle, \tag{3.7}$$

where λ is a number which is dependent on t and α. In infinitesimal form,

$$(1 + i\delta tH)\,A(1 - i\delta tH)\,|\,\alpha\rangle = (\alpha - i\delta tf)\,|\,\alpha\rangle, \tag{3.8}$$

where we have written $\lambda = \alpha - i\delta tf$, and f is independent of t. Then,

$$[A,H]\,|\,\alpha\rangle = f\,|\,\alpha\rangle\,. \tag{3.9}$$

The dependence of f on α is obtained from the relation

$$\begin{aligned} f \equiv f(\alpha) &= \langle 0\,|\,[A,H]\,\|\,\alpha\rangle \\ &= \langle 0\,|\,[A,H]\,|\,0\rangle + \sum_{n=1}^{\infty} \frac{\langle 0\,|\,[A,H]\,|\,n\rangle}{\prod_{m=1}^{n} F(m)}\,\alpha^n, \end{aligned} \tag{3.10}$$

which shows[15] that $f(\alpha)$ is analytic in α and therefore that $f(A)$ is well defined.

Sufficiency follows from the fact that Eq. (3.8) is a direct consequence of assuming Eq. (3.6)

(iii) We observe now the important fact that the set of all operators H which satisfy Eq. (3.6) for a given A constitutes a Lie algebra. For, if H_1 and H_2 are members of this set whose commutators with A are $f_1(A)$ and $f_2(A)$, respectively, it follows that

$$[A, \lambda_1 H_1 + \lambda_2 H_2] = \lambda_1 f_1(A) + \lambda_2 f_2(A), \tag{3.11}$$

and

$$[A,[H_1,H_2]] = f_2(A)\,\frac{df_1(A)}{dA} - f_1(A)\,\frac{df_2(A)}{dA}, \tag{3.12}$$

which shows that the set is linear and closed with respect to commutation. Eq. (3.12) is deduced from the Jacobi identity and the analyticity of f_1 and f_2.

It is a remarkable fact that, within the framework outlined above, it is possible to deduce the form of all

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Hermitian operators H associated in this way with a generalized destruction operator A. Further, all the A operators separate into two classes, one of which leads to Hamiltonians belonging to physically uninteresting Abelian Lie algebras, while the other contains the shift operator E as a special case and the standard annihilation operator as a limiting case. This latter class provides the physically interesting Hamiltonians.

In demonstrating these results it is convenient to work with the states $\|\alpha\rangle$ as defined in Eq. (3.5) because they permit differential representations of operators. Thus we have from Eq. (3.8)

$$(1 - i\delta t H)\,\|\,\alpha\rangle = (1 - i\delta t\, g(\alpha,\alpha^*))\,\|\,\alpha - i\delta t f(\alpha)\rangle = \left(1 - i\delta t\left(g(\alpha,\alpha^*) + f(\alpha)\frac{\partial}{\partial\alpha}\right)\right)\|\,\alpha\rangle\,. \tag{3.13}$$

The function $g(\alpha,\alpha^*)$, to be determined, serves not only to represent a possible phase factor, but also to preserve the normalization (3.5), and is therefore not necessarily real. Then,

$$H\,\|\,\alpha\rangle = [f(\alpha)\frac{\partial}{\partial\alpha} + g(\alpha,\alpha^*)]\,\|\,\alpha\rangle. \tag{3.14}$$

We must now apply the requirements of Hermiticity to H, which is done by requiring that

$$\langle\beta\,\|\,H\,\|\,\alpha\rangle = \langle\alpha\,\|\,H\,\|\,\beta\rangle^* \tag{3.15}$$

for all α and β. If we write $\|\,\alpha\rangle$ as

$$\|\,\alpha\rangle = \sum_{n=0}^{\infty} h_n \alpha^n\,|n\rangle, \tag{3.16}$$

and define the function $\psi(\zeta)$ by

$$\psi(\zeta) = \sum_{n=0}^{\infty} h_n^2 \zeta^n, \tag{3.17}$$

$\psi(\zeta)$ is analytic within the circle $|\zeta| < R^2$, where R is the spectral radius of A. The matrix element in Eq. (3.15) becomes

$$\langle\beta\,\|\,H\,\|\,\alpha\rangle = [f(\alpha)\frac{\partial}{\partial\alpha} + g(\alpha,\alpha^*)]\psi(\alpha\beta^*). \tag{3.18}$$

Setting $\beta = 0$ in Eq. (3.15) gives

$$g(\alpha,\alpha^*) = f(0)^*\, h_1^2\alpha + g(0)^*, \tag{3.19}$$

which shows that g is analytic and, in fact, linear in α and that $g(0)$ is real. Putting Eq. (3.19) into Eq. (3.18), differentiating the latter with respect to β^* and then setting $\beta^* = 0$ yields the following equation for f,

$$f(\alpha) = f_0 + f_1^*\alpha + \frac{f_0^* h_1^2}{k+1}\,\alpha^2, \tag{3.20}$$

where $f_0 = f(0)$, $f_1 = f'(0)$ and

$$1/(k+1) = (2h_2^2/h_1^4) - 1. \tag{3.21}$$

Differentiating $f(\alpha)$ once with respect to α and setting $\alpha = 0$ shows that f_1 is real. Finally, using Eq. (3.20) with the Hermiticity requirement leads to

$$(f_0\beta^* - f_0^*\alpha)\left[\psi'(\zeta) - \frac{h_1^2}{k+1}\,\zeta\psi'(\zeta) - h_1^2\psi(\zeta)\right] = 0, \tag{3.22}$$

with $\zeta = \alpha\beta^*$. A moment's reflection now shows that we must have either

(A) $f_0 = 0$

or

(B) $\psi'(\zeta) - [h_1^2/(k+1)]\zeta\psi'(\zeta) - h_1^2\psi = 0.$

Case (A) may be disposed of quickly by noting that Eq. (3.14) becomes

$$H\,\|\,\alpha\rangle = [f_1\alpha\frac{\partial}{\partial\alpha} + g(0)]\,\|\,\alpha\rangle = [f_1 N + g(0)]\,\|\,\alpha\rangle\,. \tag{3.23}$$

Once stated in general operator form, the restriction to the specific normalization defined by $\|\,\alpha\rangle$ becomes unnecessary and we see that H is a real, linear combination of N and the unit operator I. It is therefore an element of the two-dimensional Abelian Lie algebra $U(1) \times U(1)$. The transformations generated by N and I are physically uninteresting in that they merely represent the unperturbed oscillator.

The differential equation in case (B), along with the condition $\zeta(0) = 1$, has the solution

$$\psi(\zeta) = \{1 - [(h_1^2\zeta)/(k+1)]\}^{-(k+1)}. \tag{3.24}$$

In order to match this with Eq. (3.17) we note that there is always a finite circle in the ζ-plane within which we can expand Eq. (3.24) in a convergent power series

$$\psi(\zeta) = \sum_{n=0}^{\infty} \frac{\Gamma(k+n+1)}{n!\,\Gamma(k+1)}\left(\frac{h_1^2\zeta}{k+1}\right)^n. \tag{3.25}$$

Note the implicit requirement that $k > -1$ to ensure that the coefficients in the power series all be positive in conformity with the definition of Eq. (3.17). The radius of convergence of the above power series is the square of the spectral radius of A.

Taking the square root of the ratio of successive coefficients in Eq. (3.25) shows the $F(n)$ of Eq. (3.3) to be

$$F(n) = (1/h_1)\sqrt{n(k+1)/(k+n)}\,. \tag{3.26}$$

The factor $1/h_1$ is a scale factor which plays no essential role in the eigenstates of the resulting A operators. The choice $h_1 = 1$ results in $k = 0$ corresponding to the unit shift operator E, while $k \to \infty$ corresponds, as we shall see, to the annihilation operator. We refer to the resulting A as the generalized destruction operator $A(k)$, i.e.,

$$A(k) = E\sqrt{N(k+1)/(N+k)}. \tag{3.27}$$

It follows from this expression that the spectral radius of $A(k)$ is $\sqrt{k+1}$. Thus, in terms of a single real parameter k, we have the general form of the annihilation type operator associated with Case (B). It is now a simple matter to exhibit the associated eigenstates $|\alpha,k\rangle$ in normalized form. The normalization factor follows from the definition of $\psi(\zeta)$ in Eq. (3.17) with $\zeta = |\alpha|^2$ and from the expression for $\psi(\zeta)$ in Eq. (3.24). The result is

$$|\alpha,k\rangle = \left[1 - \frac{|\alpha|^2}{k+1}\right]^{(k+1)/2} \times \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+k+1)}{n!\,\Gamma(k+1)}}\left(\frac{\alpha}{\sqrt{k+1}}\right)^n |n\rangle. \tag{3.28}$$

It now remains to deduce the form of the associated Hamiltonian and to show that it is an arbitrary linear

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combination of interaction terms and the free oscillator term. Our starting point is once again Eq. (3.14), into which we substitute the expression in Eq. (3.20). This gives

$$H \| \alpha\rangle = \left[\left(f_0 + f_1\alpha + \frac{f_0^*}{k+1}\,\alpha^2\right)\frac{\partial}{\partial\alpha} + (f_0^*\alpha + g(0))\right] \| \alpha\rangle . \tag{3.29}$$

Using

$$\frac{\partial}{\partial\alpha}\| \alpha\rangle = \sqrt{\frac{N(N+k)}{k+1}}\; E^+ \| \alpha\rangle , \tag{3.30}$$

and

$$\alpha^2 \frac{\partial}{\partial\alpha}\| \alpha\rangle = \sqrt{\frac{N(N+k)}{k+1}}\, E^+ A^2 \| \alpha\rangle, \tag{3.31}$$

we are again freed of the $\| \alpha\rangle$ normalization and obtain, after some rearrangement,

$$\begin{aligned} H &= (f_0/\sqrt{k+1})H_+ + (f_0^*/\sqrt{k+1})H_- + f_1 N + g(0) \\ &= C_0 + C_1H_1 + C_2H_2 + C_3H_3, \end{aligned} \tag{3.32}$$

where

$$H_1 + iH_2 \equiv H_+ = H_-^+ = \sqrt{N(N+k)}\,E^+, \tag{3.33a}$$

$$H_3 = N + (k+1)/2 \tag{3.33b}$$

and C_0, C_1, C_2, C_3 are independent real numbers. The Hamiltonian of Eq. (3.32) is an element of the Lie algebra of $O(2,1) \times U(1)$, since H_1, H_2, and H_3 satisfy the commutation relations of $O(2,1)$. The basic equation (3.6) takes the form

$$\begin{aligned} &[A, H_3] = A, \\ &[A, H_+] = \sqrt{k+1}, \\ &[A, H_-] = A^2/(k+1)^{1/2}. \end{aligned} \tag{3.34}$$

4. DYNAMICS OF THE $|\alpha, k\rangle$ STATES

In this section we turn our attention to the group of transformations induced on the $|\alpha, k\rangle$ states by unitary operators e^{-itH}, where H is a Hamiltonian of the form (3.32). Since the eigenvalue transformations are unaffected by constant terms in the Hamiltonian, we restrict ourselves to the operators H_1, H_2, H_3 by choosing $g(0) = f_1[(k+1)/2]$. The resulting Hamiltonian will be referred to as $H(k)$, i.e.,

$$H(k) = (1/\sqrt{k+1})(f_0H_+ + f_0^*H_-) + f_1H_3. \tag{4.1}$$

Using the pertinent results of the previous section, we can rewrite Eq. (3.13) as

$$(1 - i\delta tH(k))\,\| \alpha, k\rangle = (1 - i\delta tg(\alpha))\,\| \alpha - i\delta tf(\alpha), k\rangle. \tag{4.2}$$

It then follows that

$$\begin{aligned} e^{-itH(k)}\| \alpha, k\rangle &= \lim_{n\to\infty}\left(1 - i\frac{t}{n}\,H(k)\right)^n \| \alpha, k\rangle \\ &= \exp\{-i\int_0^t dt\, g[\alpha(t)]\}\,\| \alpha(t), k\rangle, \end{aligned} \tag{4.3}$$

where $\alpha(t)$ satisfies the Riccati type[16] differential equation

$$\frac{d\alpha(t)}{dt} = -if[\alpha(t)] = -i\left[f_0 + f_1\alpha(t) + \frac{f_0^*}{k+1}\,\alpha^2(t)\right], \tag{4.4}$$

with initial condition $\alpha(0) = \alpha$.

The trivial case $f_0 = 0$ represents the free oscillator with the obvious result that $\alpha(t) = \alpha e^{-if_1t}$. For $f_0 \neq 0$, we make the substitution

$$\alpha(t) = -i\frac{k+1}{f_0^*}\,\frac{1}{w(t)}\,\frac{dw(t)}{dt}. \tag{4.5}$$

The function $w(t)$ need be determined only to within a multiplicative constant. This can be done by noting that it satisfies the second order differential equation

$$\frac{d^2w}{dt^2} + if_1\frac{dw}{dt} - \frac{|f_0|^2}{k+1}\,w = 0. \tag{4.6}$$

Imposing the condition $\alpha(0) = \alpha$ on the general solution of this equation gives

$$w(t) = e^{-if_1t/2}\left[\cos st + i\left(\frac{f_1}{2} + \frac{f_0^*\alpha}{k+1}\right)\frac{\sin st}{s}\right], \tag{4.7}$$

where

$$s = \begin{cases} \left(\dfrac{f_1^2}{4} - \dfrac{|f_0|^2}{k+1}\right)^{1/2}, & \dfrac{f_1^2}{4} > \dfrac{|f_0|^2}{k+1}, \\ i\left(\dfrac{|f_0|^2}{k+1} - \dfrac{f_1^2}{4}\right)^{1/2}, & \dfrac{f_1^2}{4} < \dfrac{|f_0|^2}{k+1}. \end{cases} \tag{4.8}$$

This in turn leads to the results that $\alpha/\sqrt{k+1}$ undergoes the linear fractional transformation

$$\alpha(t)/\sqrt{k+1} = \frac{a(\alpha/\sqrt{k+1}) + b}{b^*(\alpha/\sqrt{k+1}) + a^*} \tag{4.9}$$

with

$$a = \cos st - i(f_1/2)(\sin st/s), \tag{4.10a}$$

and

$$b = -(if_0/\sqrt{k+1})(\sin st/s). \tag{4.10b}$$

Thus $|a|^2 - |b|^2 = 1$, so that the transformation has unit determinant.[17] It can be verified also that it maps the interior of the spectral circle $|\alpha| < \sqrt{k+1}$ into itself.

Having established the transformation properties of the eigenvalues within the spectral circle, it now remains to evaluate the multiplier $\exp\{-i\int_0^t dtg[\alpha(t)]\}$ in Eq. (4.3), which, it will be recalled, does not generally have unit magnitude because it is defined with respect to the $\| \alpha, k\rangle$. From Eq. (4.5) and the expression for $g(\alpha)$, it follows immediately that

$$\int_0^t dtg[\alpha(t)] = +\frac{k+1}{2}\,f_1t - i(k+1)\int_0^t dt\,\frac{1}{w(t)}\,\frac{dw(t)}{dt}, \tag{4.11}$$

which leads to the multiplier

$$\begin{aligned} &\exp\{-i\int_0^t dtg[\alpha(t)]\} \\ &\quad = \exp\left\{-(k+1)\ln\left[\cos st + i\left(\frac{f_1}{2} + \frac{f_0^*\alpha}{k+1}\right)\frac{\sin st}{s}\right]\right\} \end{aligned} \tag{4.12}$$

The branch of the logarithm in the exponent is determined by taking the principal value zero at $t = 0$, then demanding continuity in t.

The results obtained thus far in this section are summarized in the formula

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$$\exp\left\{-it\left(\frac{f_0 H_+ + f_0^* H_-}{\sqrt{k+1}} + f_1 H_3\right)\right\} \| \alpha, k \rangle$$
$$= \exp\left\{-(k+1)\ln\left[\cos st + i\left(\frac{f_1}{2} + \frac{f_0^*\alpha}{k+1}\right)\frac{\sin st}{s}\right]\right\}$$
$$\times \left\| \sqrt{k+1} \quad \frac{a\alpha + b\sqrt{k+1}}{b^*\alpha + a^*\sqrt{k+1}}, k \right\rangle, \tag{4.13}$$

where a, b, and s are given by Eqs. (4.8) and (4.10).

We are now in a position to exhibit unitary transformations which transform the ground state to an arbitrary (normalized) state $|\alpha, k\rangle$. Setting $\alpha(0) \equiv \alpha = 0$ in Eq. (4.9), it is apparent from Eq. (4.10) that an arbitrary $\alpha(t)$ is easily attained by making the choice $f_1 = 0$. For we then have

$$\frac{\alpha(t)}{\sqrt{k+1}} = \frac{b}{a^*} = -\frac{if_0}{|f_0|}\tanh\left(\frac{|f_0|t}{\sqrt{k+1}}\right) \tag{4.14}$$

and

$$\exp\left\{-i\int_0^t dt\, g[\alpha(t)]\right\} = \left[\cosh\left(\frac{|f_0|t}{\sqrt{k+1}}\right)\right]^{-(k+1)}$$
$$= \left[1 - \tanh^2\left(\frac{|f_0|t}{\sqrt{k+1}}\right)\right]^{(k+1)/2}. \tag{4.15}$$

Looking at Eq. (3.28) we see that (4.14) provides just the proper normalization factor, as it must, since $\| 0\rangle = |0\rangle$. The resulting unitary operator can be put into a particularly simple form which does not depend explicitly on the parameter t by choosing

$$f_0 = ie^{i\phi}, \quad t = \rho\sqrt{k+1}. \tag{4.16}$$

This leads to the relations

$$D_k(\rho, \phi)\,|0\rangle = |\, e^{i\phi}\sqrt{k+1}\tanh\rho\rangle, \tag{4.17}$$

where the unitary operator $D_k(\rho, \phi)$ is defined by[18]

$$D_k(\rho, \phi) \equiv \exp\{\rho(e^{i\phi}H_+ - e^{-i\phi}H_-)\}. \tag{4.18}$$

The states $|z\rangle$ of Eq. (3.1) are obtained by setting $k = 0$, so that

$$D_0(\rho, \phi) = \exp\{\rho(e^{i\phi}NE^+ - e^{-i\phi}EN)\}$$
$$= \exp\{\rho(e^{i\phi}\sqrt{N}\,a^+ - e^{-i\phi}a\sqrt{N})\}, \tag{4.19}$$

and

$$D_0(\rho, \phi)|0\rangle = |\, z\rangle, \quad z = e^{i\phi}\tanh\rho. \tag{4.20}$$

These results, and those of the previous section, provide the operators called for by the arguments of Sec. 2. In particular, we note that the interaction Hamiltonians which generate phase states are linear in $a\sqrt{N}$ and $\sqrt{N}\,a^+$, but not in q and p.

An interesting property of the Hamiltonians (4.1) appears when we consider the eigenvalue transformations of Eq. (4.9) in more detail. The nature of the transformations depends on the value of the parameter s. For each value of k, the Hamiltonians (4.1) fall into two classes according to whether the parameters f_0 and f_1 are such that s is real or pure imaginary as indicated in Eq. (4.8).

The difference between the two classes is best understood if we consider the points which are invariant under the transformation of eigenvalues induced by each Hamiltonian.[19] These can be determined either directly from Eq. (4.9), or by setting the right-hand side of the differential equation (4.4) equal to zero. There are generally two such points, given by

$$\alpha = (k+1)(-f_1 \pm 2s)/2f_0^*. \tag{4.21}$$

It is not difficult to see that for the first class of Hamiltonians (s real), both invariant points lie on the same ray from the center of the spectral circle, one inside the circle and one outside. For the other class (s pure imaginary), both points lie on the circumference of the circle. For the case $s = 0$, which properly belongs to the second class, the two invariant points coincide, and lie on the circumference.

The eigenvalue transformations induced by the first class of Hamiltonians are periodic in t with period π/s. The eigenvalue α, starting from any initial value inside the spectral circle, follows a closed trajectory which circumscribes the invariant point within the circle.[20] Thus, the free oscillator mean energy, related to $|\alpha|$, varies periodically with t for these Hamiltonians.

As $s \to 0$, the trajectories are pinched between the two coalescing invariant points, so that in the limit, all trajectories pass through this single point, which now lies on the circumference of the spectral circle. At the same time, the period becomes infinite.

This behavior implies that the asymptotic state of the system for $t \to \pm\infty$ is independent of the initial state, and is determined only by the parameters f_0 and f_1 that specify the Hamiltonian.[21] This final state has infinite mean energy and sharp phase resolution. A Hamiltonian of this class therefore leaves its imprint on the system in the value of the phase of its final state.

If the parameter s is pure imaginary, the trajectories pass through both invariant points on the circumference of the circle $|\alpha| = \sqrt{k+1}$. There are then two asymptotic states of infinite energy, one for $t \to +\infty$ and one for $t \to -\infty$. The behavior is otherwise similar to the case $s = 0$.

It is intuitively obvious from Eq. (3.27) that the limit $k \to \infty$ should lead to the well-known coherent states

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{4.22}$$

This follows rigorously from the fact that, for a given α, we can choose k sufficiently large so that α is within the spectral radius of $A(k)$ and then compute $\| |\alpha\rangle - |\alpha, k\rangle \|^2$. Noting that $\langle\alpha\,|\alpha, k\rangle$ is real,

$$\| \,|\alpha\rangle - |\,\alpha, k\rangle\,\|^2 = 2(1 - \langle\alpha\,|\,\alpha, k\rangle), \tag{4.23}$$

where

$$\langle\alpha\,|\,\alpha, k\rangle = e^{-|\alpha|^2/2}\left(1 - \frac{|\alpha|^2}{k+1}\right)^{(k+1)/2}$$
$$\times \sum_{n=0}^{\infty}\left(\frac{\Gamma(n+k+1)}{\Gamma(k+1)(k+1)^n}\right)^{1/2}\frac{|\alpha|^{2n}}{n!}. \tag{4.24}$$

Since

$$\frac{\Gamma(n+k+1)}{\Gamma(k+1)(k+1)^n} \geq 1, \tag{4.25}$$

we have

$$e^{|\alpha|^2/2}\left(1 - \frac{|\alpha|^2}{k+1}\right)^{(k+1)/2} \leq \langle\alpha\,|\,\alpha, k\rangle \leq 1, \tag{4.26}$$

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so that

$$\langle \alpha \mid \alpha, k\rangle \xrightarrow[k\to\infty]{} 1, \tag{4.27}$$

which proves the point.

Similarly, the behavior of the unitary operator $D_k(\rho,\phi)$ defined by Eq. (4.17) is of interest. Using essentially heuristic arguments, we see[22] that for a fixed

$$\alpha = e^{i\phi}\sqrt{k+1}\,\tanh\rho, \tag{4.28}$$

ρ must become small as $k \to \infty$. Thus $\alpha \simeq e^{i\phi}\sqrt{k}\,\rho$, and from (3.33a),

$$D_k(\rho,\phi) \to \exp\left\{\frac{\alpha}{\sqrt{k}}\sqrt{kN}\,E^{+} - \frac{\alpha^*}{\sqrt{k}}\,E\sqrt{kN}\right\} = e^{\alpha a^{+} - \alpha^* a}, \tag{4.29}$$

which is just Glauber's[10] $D(\alpha)$ operator.

These results encourage us to seek a resolution of the identity of the form

$$I = \int d^2\alpha\,\rho(\mid\alpha\mid^2, k)\mid\alpha, k\rangle\langle\alpha, k\mid, \tag{4.30}$$

with the weighting function $\rho(\mid\alpha\mid^2, k)$ to be determined, and $d^2\alpha = d(\mathrm{Re}\alpha)\,d(\mathrm{Im}\alpha)$ with integration over the circle $\mid\alpha\mid < \sqrt{k+1}$. By introducing polar coordinates in the α-plane and making the change of variable $t = \mid\alpha\mid^2/k+1$, the integral in (4.30) can be written as

$$\int d^2\alpha\,\rho(\mid\alpha\mid^2, k)\mid\alpha, k\rangle\langle\alpha, k\mid = \pi\sum_{n=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(n+1)\Gamma(k)}\int_0^1 dt\,\bar{\rho}(t,k)(1-t)^{k+1}t^n\mid n\rangle\langle n\mid, \tag{4.31}$$

where $\rho(\mid\alpha\mid^2, k) = \bar{\rho}(t,k)$. A resolution of the identity is obtained if

$$\int_0^1 dt\,\bar{\rho}(t,k)(1-t)^{k+1}t^n = \frac{1}{\pi}\,\frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k+1)}. \tag{4.32}$$

Using

$$\int_0^1 dt\,(1-t)^{q-1}t^{p-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{4.33}$$

when the real parts of p and q are positive, we see that

$$\bar{\rho}(t,k) = (1/\pi)\,[1/(1-t)^2], \tag{4.34a}$$

or,

$$\rho(\mid\alpha\mid^2, k) = (1/\pi)\,[1/(1-\mid\alpha\mid^2/(k+1))^2], \tag{4.34b}$$

with the condition $k > 0$. This gives the result

$$\frac{1}{\pi}\int d^2\alpha\left(1-\frac{\mid\alpha\mid^2}{k+1}\right)^{-2}\mid\alpha, k\rangle\langle\alpha, k\mid = I, (k>0). \tag{4.35}$$

Note that the weighting function has sufficiently singular behavior for its integral to diverge, a property which is to be expected since $\mathrm{Tr}(I) = \infty$. The condition $k > 0$ indicates that this resolution fails for the phase states (3.1). This is a particularly vexing fact in view of the physical interest attached to these states. The reason for it is most simply seen from Eq. (4.32), which shows that the condition $k = 0$ demands that all of the moments of the function $\bar{\rho}(t,0)(1-t)$ on the unit interval be equal. This in turn forces $\bar{\rho}(t,0)$ to have δ-function type behavior at $t = 1$. In a crude sense, this indicates that the unit shift operator E tries very hard to behave like a unitary operator.

5. GENERALIZATIONS AND EXTENSIONS

It should be apparent by now that the formalism developed thus far already exceeds in generality the requirements posed in Sec. 2. Quite aside from relevance to phase states there is thus the mathematically intriguing question of whether the methods developed here are useful for the study of groups.

The essential feature of our procedure is the association of a destruction operator with a unitary group in such a fashion that the eigenstates of the operator transform into themselves under the action of the group. Because eigenstates of destruction operators are of necessity infinite dimensional, this leads to infinite dimensional representations of the group. However, there is a limiting sense in which the association between destruction operator, eigenstate, and group carries over into finite dimensional spaces. In this section we illustrate this by extending our formalism to include representations of $O(3)$ as well as $O(2,1)$.

Our starting point is a consideration of the effect of removing the restriction $-1 < k < \infty$ from the parameter k by extending it to the complex plane. The operators $A(k)$ of Eq. (3.27) and the H of Eq. (3.33) remain well defined. Their matrix elements on the number basis are in fact analytic in the complex k-plane cut from -1 to $-\infty$. Further, the $\mid\alpha, k\rangle$ of Eq. (3.28) continue to be normalized eigenstates of $A(k)$ for $\mid\alpha\mid < \sqrt{\mid k+1\mid}$.

The Hermiticity of the H, which is of course generally lost in this process, can be restored for negative integral k (approached, for example, from the upper half-plane) by a simple modification based on the fact that the H now reduce the oscillator space to two invariant subspaces. Thus, let $k = -p$, where p is a positive integer greater than unity. The Eq. (3.33a) shows that

$$H_{+}\mid p-1\rangle = 0 \tag{5.1a}$$

and

$$H_{-}\mid p\rangle = 0. \tag{5.1b}$$

Therefore, a natural division of the number basis into two invariant bases is defined. The vectors $\mid n\rangle$ with $0 \le n \le p-1$ span the p-dimensional subspace X_p, while those with $n \ge p$ span the infinite dimensional subspace X_p^{∞}. In the latter subspace the H remain Hermitian as defined, and are in fact identical, except for a relabeling of the basic states, with those for the case $k = +p$.

We therefore fix our attention on the subspace X_p, where multiplication of $H_{\pm}$ by $-i$ achieves the desired result. The re-defined Hamiltonians,

$$H_1 + iH_2 \equiv H_{+} = H_{-}^{\dagger} = \sqrt{N(p-N)}\,E^{+} \tag{5.2a}$$

and

$$H_3 = N - (p-1)/2 \tag{5.2b}$$

satisfy the $O(3)$ commutation rules. We have thus been led naturally to the p-dimensional representation of the $O(3)$ algebra. In conventional notation, $p = 2j+1$ and $J_3 = N - j$.

The question of obvious interest at this point is the status of the $\mid\alpha, k\rangle$ states. Inspection of Eq. (3.28) shows that, as k approaches $-p$ from the upper half of the complex plane, the coefficients of $\mid\alpha, k\rangle$ in X_p^{∞} becomes vanishingly small. In the limit, a well-defined p-dimensional state $\mid\alpha, p\rangle$ is obtained:

$$\mid\alpha, p\rangle = \left(1+\frac{\mid\alpha\mid^2}{p-1}\right)^{-(p-1)/2}\sum_{n=0}^{p-1}\sqrt{\binom{p-1}{n}}\left(\frac{\alpha}{\sqrt{p-1}}\right)^n\mid n\rangle, \tag{5.3}$$

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where

$$\binom{p-1}{n} = \frac{(p-1)!}{n!(p-1-n)!} \tag{5.4}$$

is the standard binomial coefficient. The state $|\alpha,p\rangle$ is now defined for all α, and not merely those within some spectral radius, as it is when k is not a negative integer. Note that $A(k)$, which now becomes

$$A_p \equiv A(-p) = E\sqrt{\frac{N(p-1)}{p-N}}, \tag{5.5}$$

is well defined on X_p. Also, its commutators with the H of (5.2) satisfy the basic relation (3.6). However, the $|\alpha,p\rangle$ of (5.3) are no longer eigenstates of A in the strict sense in view of the fact that a destruction operator cannot have eigenstates in a finite dimensional space. There is, nevertheless, a limiting sense in which the $|\alpha,p\rangle$ are eigenstates of A_p. This can be seen from the equation $A(k)\,|\,\alpha,k\rangle = \alpha\,|\,\alpha,k\rangle$, which holds rigorously for k indefinitely close to $-p$. We therefore have

$$\lim_{k\to -p} A(k)\,|\,\alpha,k\rangle = \alpha\,|\,\alpha,p\rangle, \tag{5.6}$$

which, however, does not imply $A_p\,|\,\alpha,p\rangle = \alpha\,|\,\alpha,p\rangle$. In effect, the singular limiting behavior of $A(k)$ on the state $|\,p\rangle$ combines with the vanishingly small projection of $|\alpha,k\rangle$ on $|\,p\rangle$ to give a finite amplitude on $|\,p-1\rangle$. This feature is not present when A is restricted to X_p.

The above remarks suggest the retention of the concept of eigenstate of $A(k)$ even in the limit, and we thus refer to the $|\alpha,p\rangle$ as extended eigenstates of $A(k)$. We now show that these states also retain the property of transforming among themselves under transformations generated by the Hamiltonians of Eq. (5.2).

Our demonstration is based on the fact that the H can be represented as differential operators when acting on the nonnormalized states $\|\alpha,p\rangle$. The following equations are easily verified from (5.2) and (5.3):

$$H_3\,\|\alpha,p\rangle = \left(\alpha\frac{\partial}{\partial\alpha} - \frac{p-1}{2}\right)\|\alpha,p\rangle, \tag{5.7a}$$

$$H_+\,\|\alpha,p\rangle = \sqrt{p-1}\,\frac{\partial}{\partial\alpha}\,\|\alpha,p\rangle, \tag{5.7b}$$

$$H_-\,\|\alpha,p\rangle = \left(\sqrt{p-1}\,\alpha - \frac{1}{\sqrt{p-1}}\,\alpha^2\frac{\partial}{\partial\alpha}\right)\|\alpha,p\rangle. \tag{5.7c}$$

The differential relations imply that one can immediately employ the methods of Sec. 4, beginning with Eq. (4.2). In fact, one obtains, "mutatis mutandis", as the analog of Eq. (4.13), the result

$$\exp\{-it[(p-1)^{-1/2}(f_0H_+ + f_0^*H_-) + f_1H_3]\}\,\|\alpha,p\rangle$$
$$= \exp\left\{(p-1)\ln\left[\cos st + i\left(\frac{f_1}{2} - \frac{f_0^*\alpha}{p-1}\right)\frac{\sin st}{s}\right]\right\}$$
$$\times\,\|\sqrt{p-1}\,\frac{a\alpha + ib\sqrt{p-1}}{ib^*\alpha + a^*\sqrt{p-1}}, p\rangle, \tag{5.8}$$

where

$$s = \left(\frac{f_1^2}{4} + \frac{|f_0|^2}{p-1}\right)^{1/2} > 0, \tag{5.9}$$

and

$$a = \cos st - i\frac{f_1}{2}\frac{\sin st}{s}, \tag{5.10a}$$

$$b = -\frac{f_0}{\sqrt{p-1}}\frac{\sin st}{s}. \tag{5.10b}$$

The analog of Eqs. (4.17) and (4.18) is

$$D_p(\rho,\phi)\,|\,0\rangle = |\sqrt{p-1}\,e^{i\phi}\tanh\rho, p\rangle, \tag{5.11a}$$

$$D_p(\rho,\phi) = \exp\{\rho e^{i\phi}H_+ - \rho e^{-i\phi}H_-\}. \tag{5.11b}$$

With these formulas, our extension of the formalism to $O(3)$ is complete.

We can characterize the results of this section in group theoretical terms by saying that we pass continuously from unitary representations of $O(2,1)$ to the unitary representations of $O(3)$ via a continuum of nonunitary representations of $O(2,1)$. Thus, representations of the compact group appear at isolated values of the parameter which labels the representations of the noncompact group.

6. REPRESENTATIONS OF $O(3)$ AND $O(2,1)$

We have shown in the preceding sections that the groups underlying the dynamics of phase states are $O(2,1)$ and $O(3)$. These groups are realized in our formalism in two distinct way: (i) as mappings of the eigenvalues of the generalized destruction operator, and (ii) as transformations of the Hilbert space of harmonic oscillator states. The transformations involved are familar.[23-25] In the case of $O(2,1)$, the eigenvalue mappings are the linear fractional transformations obtained by stereographic projection of the unit hyperboloid onto the complex plane. The group of such transformations is known to be isomorphic with $O(2,1)$, and can therefore be identified with this group. A similar situation obtains with respect to $O(3)$ and projections of the unit sphere. The spectrum of the generalized destruction operator can therefore be regarded as the supporting space of the underlying groups $O(2,1)$ and $O(3)$. The transformations of Hilbert space on the other hand are simply the linear irreducible representations of the two groups.

The generalized destruction operator brings the supporting spaces of the group and of its representations into particularly close association. It is an operator which is defined on the representation space and whose spectrum serves as the space on which the group itself acts. Further, the mapping of the spectrum is associated with a mapping of eigenstates by the group, and thereby determines the transformation in Hilbert space associated with a particular group element. It will be shown in a forthcoming paper that these features are general and apply to groups other than $O(3)$ and $O(2,1)$. Here, we content ourselves with showing that some of the familiar results pertaining to representations of the $O(2,1)$ and $O(3)$ groups appear in the present context.

We begin by identifying the representations that have been obtained. In the case of $O(2,1)$, irreducible representations are characterized[25] by the value of the Casimir operator $Q = -H_1^2 - H_2^2 + H_3^2$ and the spectrum of the generator H_3. These are found from Eq. (3.33) in our case:

$$Q = \tfrac{1}{4}(k^2-1), \tag{6.1}$$

$$H_3 = n + \tfrac{1}{2}(k+1), \quad n = 0,1,2,\cdots. \tag{6.2}$$

This infinite family of representations, labeled by the

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real parameter k, coincides with the representations $D^{-}(\Phi)$ of Ref. (25), with $\Phi = -\frac{1}{2}(k+1)$. The representations $D^{+}(\Phi)$ are also obtained in our formalism by a different choice of labeling for the generators, which corresponds to the substitutions $H_3 \to -H_3$ and $H_2 \to -H_2$. These are multivalued representations of $O(2,1)$ unless k is an odd integer. Our basic requirement associating a generalized destruction operator with a representation therefore leads to all the unitary representations of $O(2,1)$ in which the spectrum of H_3 is bounded either from above or from below. The remaining representations, in which H_3 has an unbounded spectrum, cannot be supported on the harmonic oscillator basis. They can be obtained, however, by our method if an extension of the basis to negative n is made. The details of this extension will be given in a future paper.

The case of $O(3)$ is simpler. It is evident from the remarks following Eq. (5.2) and from the allowed values of p that we obtain all of the unitary representations except for the trivial one belonging to the eigenvalue zero of the Casimir operator $Q = J^2 = H_1^2 + H_2^2 + H_3^2$. Thus the generalized destruction operator and its eigenstates bring into association the unitary representations of $O(3)$ and the bounded unitary representations of $O(2,1)$.

If we now turn our attention to the eigenstates (3.28) and (5.3), we find that they have a simple group theoretic significance: they are the transforms under the operators of the group of the single state $|0\rangle$. The existence of the operators $D_k(\rho,\phi)$ and $D_p(\rho,\phi)$ of Eqs. (4.16) and (5.11) guarantees that all the eigenstates can be obtained from the ground states in this way, and the basic property (3.6) insures that we obtain only eigenstates.

In the case of the rotation group, the ground state corresponds, for a given representation $j = \frac{1}{2}(p-1)$, to the state with $H_3 = -j$, i.e., the state with "spin down" with respect to the 3-axis. The rotation which transforms this state into another extended eigenstate of the generalized destruction operator simply rotates this spin to some other direction in space. We may therefore characterize the eigenstates as those states having the minimum value of the component of angular momentum in some definite direction.

A similar characterization is possible for $O(2,1)$ if we interpret H_1 and H_2 as the generators of pure Lorentz transformations in two orthogonal spatial directions, and H_3 as the generator of rotations in the plane of these directions. Here again, although the eigenvalues of H_3 are no longer restricted to integers or half-integers, the state $|0\rangle$ corresponds to the minimum eigenvalue. Therefore the eigenstates of the generalized destruction operator are just those states which have minimum eigenvalue of H_3 in some definite Lorentz frame.

Another interesting description of the eigenstates is obtained by considering their components $\langle n \mid \alpha\rangle$ with respect to the number basis. The above remarks indicate that these components can be written in the form $\langle n \mid D \mid 0\rangle$ for some operator D belonging to the representation. Furthermore, the quantities $\langle n \mid D \mid 0\rangle$ for an arbitrary operator of the representation form the components of some eigenstate of the generalized destruction operator. This shows that the first columns of the matrix representatives of all the operators of the representation comprise the class of eigenvectors of the generalized destruction operator.

The full matrix for an operator D of exponential form is also calculable from the eigenstates of the generalized destruction operator. The results are not new and the derivation is similar to treatments found in the literature,[24] but the method given here offers a considerable conceptual simplification. Beginning from Eqs. (4.13) and (5.8), we obtain an equation

$$\langle n \mid D \parallel \alpha\rangle = \exp\{-i\int_0^t dt\, g[\alpha(t)]\}\langle n \parallel \alpha(t)\rangle \qquad (6.3)$$

in which the left-hand side is expressible as a convergent power series in α with coefficients proportional to $\langle n \mid D \mid n'\rangle$, and the right-hand side is a known analytic function of α for a given D. We can therefore obtain $\langle n \mid D \mid n'\rangle$ by comparison of coefficients on either side of Eq. (6.3). In this way we can obtain, for example, the familiar result[26] for the matrix elements of $\exp(-i\beta H_2)$ for $O(3)$, and its analog for the bounded representations $D^{-}(\Phi)$ of $O(2,1)$:

$$\begin{aligned}\langle n \mid e^{-i\psi H_2(k)} \mid n'\rangle = {} & \left(\frac{n'!\,\Gamma(n+k+1)}{n!\,\Gamma(n'+k+1)}\right)^{1/2} \\ & \times \sum_r \frac{n!}{(n'-r)!\,(n-n'+r)!} \\ & \times (-1)^{n-n'+r}\,\frac{\Gamma(n+k+r+1)}{r!\,\Gamma(n+k+1)} \\ & \times (\cosh\tfrac{1}{2}\psi)^{n-n'-k-1-2r}(\sinh\tfrac{1}{2}\psi)^{n-n'+2r}. \end{aligned} \qquad (6.4)$$

The summation over the integer r runs over a finite range which is determined by the factorials in the denominator.

Note added in proof: Since submitting this paper it has come to our attention that some properties of the states of Eq. (5.3) have been discussed in other contexts by various authors.[27]

APPENDIX

We show here that when an oscillator is in a state $|z\rangle$ of the form (3.1) with z positive real, i.e., $\phi = 0$ in Eq. (2.11), then $\delta q/\langle q\rangle$ approaches a finite number less than unity and δp becomes vanishingly small as $\langle N\rangle \to \infty$.

It is convenient to normalize units so that

$$q = (2)^{-1/2}(a + a^{+}), \qquad p = [-i(2)^{-1/2}](a - a^{+}). \qquad (A1)$$

In terms of the unit shift operator E of Eq. (2.10) we have

$$a = E\sqrt{N} = \sqrt{N+1}\,E \qquad (A2)$$

and

$$a^2 = \sqrt{N+1}\,E\sqrt{N+1}\,E = \sqrt{(N+1)(N+2)}\,E^2. \qquad (A3)$$

Using the fact that $E \mid z\rangle = z \mid z\rangle$ and z is real gives

$$\begin{aligned}\frac{\langle q^2\rangle}{\langle q\rangle^2} = {} & \frac{1}{2\langle\sqrt{N+1}\rangle^2} \\ & \times\left(\frac{(\langle N\rangle + \frac{1}{2})(\langle N\rangle + 1)}{\langle N\rangle} + \langle\sqrt{(N+1)(N+2)}\rangle\right), \end{aligned} \qquad (A4)$$

where $\langle N\rangle$ is related to z by Eq. (2.11). We shall see that the behavior of this expression depends strongly on the behavior of the expectation value

$$\langle\sqrt{N+1}\rangle = (1-z^2)\sum_{n=0}^{\infty}\sqrt{n+1}\;z^{2n}, \qquad (A5)$$

which we now examine.

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Let

$$z^2 = \frac{\langle N\rangle}{\langle N\rangle + 1} = \zeta, \tag{A6}$$

so that

$$\langle\sqrt{N+1}\rangle = \sum_{n=0}^{\infty} (\sqrt{n+1} - \sqrt{n})\,\zeta^n. \tag{A7}$$

Noting that

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{2}\int_0^1 \frac{dr}{\sqrt{n+r}}, \tag{A8}$$

enables us to write

$$\langle\sqrt{N+1}\rangle = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{dx}{x^2}\,\frac{1-e^{-x^2}}{1-\zeta e^{-x^2}}, \tag{A9}$$

where we have used

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-(n+r)x^2} = \frac{1}{\sqrt{n+r}}\ . \tag{A10}$$

Expressing ζ in terms of $\langle N\rangle$ and making the change of integration variable to $y = x\sqrt{\langle N\rangle}$ puts (A9) into the form

$$\langle\sqrt{N+1}\rangle = \frac{\sqrt{\langle N\rangle}\,(1+\langle N\rangle)}{\sqrt{\pi}}\int_0^{\infty}\frac{dy}{y^2}\,\frac{1-e^{-y^2/\langle N\rangle}}{1+\langle N\rangle\,(1-e^{-y^2/\langle N\rangle})}. \tag{A11}$$

For very large $\langle N\rangle$ the integrand in (A11) goes as $[\langle N\rangle(1+y^2)]^{-1}$. This suggests writing $\langle\sqrt{N+1}$ as

$$\langle\sqrt{N+1}\rangle = \frac{1+\langle N\rangle}{\sqrt{\pi\langle N\rangle}}\int_0^{\infty}\frac{dy}{1+y^2}\,G\left(y^2, \frac{1}{\langle N\rangle}\right), \tag{A12}$$

where the function $G(x,\epsilon)$ is defined by

$$G(x,\epsilon) = [1 + (1/x)]/[1 + (1/f(x,\epsilon))] \tag{A13}$$

and

$$f(x,\epsilon) = (1 - e^{-\epsilon x})/\epsilon. \tag{A14}$$

It may be verified by straightforward calculation that $G(x,\epsilon)$ decreases monotonically from the value $G(0,\epsilon) = 1$ to $G(\infty,\epsilon) = 1/(1+\epsilon)$. This then gives

$$\tfrac{1}{2}\sqrt{\pi\langle N\rangle} \le \langle\sqrt{N+1}\rangle \le \tfrac{1}{2}\sqrt{\pi\langle N\rangle}\,(1 + 1/\langle N\rangle). \tag{A15}$$

Going back to Eq. (A4) and using

$$\langle N\rangle + \tfrac{1}{2} < \langle\sqrt{(N+1)(N+2)}\rangle < \langle N\rangle + \tfrac{3}{2}, \tag{A16}$$

which follows directly from the fact that

$$\langle\sqrt{(N+1)(N+2)}\rangle = (1-\zeta)\sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}\,\zeta^n, \tag{A17}$$

with the appropriate inequalities holding term by term, we get

$$\frac{4}{\pi}\left(\frac{1+(1/2\langle N\rangle)}{1+(1/\langle N\rangle)}\right)^2 < \frac{\langle q^2\rangle}{\langle q\rangle^2} < \frac{4}{\pi}\left(1 + \frac{3}{2\langle N\rangle} + \frac{1}{4\langle N\rangle^2}\right). \tag{A18}$$

Thus as $\langle N\rangle$ becomes indefinitely large,

$$\frac{(\delta q)^2}{\langle q\rangle^2} = \frac{\langle q^2\rangle}{\langle q\rangle^2} - 1 \sim \frac{4}{\pi} - 1, \tag{A19}$$

or

$$\frac{\delta q}{\langle q\rangle} \sim 0.52. \tag{A20}$$

Now, since $\langle p\rangle = 0$ for a state $|z\rangle$ with $\phi = 0$, the calculation of δp involves only the calculation of $\langle p^2\rangle$, which from Eqs. (A1) and (A3) is given by

$$\langle p^2\rangle = \langle N\rangle + \tfrac{1}{2} - \left(\frac{\langle N\rangle}{\langle N\rangle+1}\right)\langle\sqrt{(N+1)(N+2)}\rangle \tag{A21}$$

The demonstration that this expression becomes vanishingly small for large $\langle N\rangle$ requires a somewhat tighter inequality than that of Eq. (A16). Such an inequality is

$$\langle\sqrt{(N+1)(N+2)} > \langle N\rangle + \tfrac{3}{2} - \tfrac{1}{4}\left\langle\frac{1}{N+\frac{3}{2}}\right\rangle, \tag{A22}$$

which again follows from the fact that it holds term by term if one writes out the expectation value as in Eq. (A17). Thus,

$$\langle p^2\rangle < \langle N\rangle + \tfrac{1}{2} - \left(1 + \frac{1}{\langle N\rangle}\right)^{-1}\left(\langle N\rangle + \tfrac{3}{2} - \tfrac{1}{4}\left\langle\frac{1}{N+\frac{3}{2}}\right\rangle\right). \tag{A23}$$

Using the obvious fact that the expectation value $\langle 1/(N+\frac{3}{2})\rangle$ is bounded from above by unity, we see that for $\langle N\rangle \gg 1$

$$\langle p^2\rangle < \frac{1}{4}\left\langle\frac{1}{N+\frac{3}{2}}\right\rangle + 0\left(\frac{1}{\langle N\rangle}\right). \tag{A24}$$

The relevant expectation value is

$$\left\langle\frac{1}{N+\frac{3}{2}}\right\rangle = (1-\zeta)\sum_{n=0}^{\infty}\frac{\zeta^n}{n+\frac{3}{2}} < \frac{1-\zeta}{\zeta}\sum_{n=0}^{\infty}\frac{\zeta^{n+1}}{n+1}, \tag{A25}$$

so that

$$\left\langle\frac{1}{N+\frac{3}{2}}\right\rangle < -\frac{1-\zeta}{\zeta}\ln(1-\zeta) = \frac{\ln(\langle N\rangle+1)}{\langle N\rangle} \xrightarrow[\langle N\rangle\to\infty]{} 0. \tag{A26}$$

We see, then, that $\langle p^2\rangle$, and thus δp, becomes vanishingly small as $\langle N\rangle$ becomes indefinitely large.

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*A preliminary report of this work and its possible group theoretic implications is contained in Y. Aharonov, H. W. Huang, J. M. Knight, and E. C. Lerner, Nuovo Cimento Lett. **2**, 1317 (1971). Parts of this work are contained in a doctoral dissertation presented by one of us (HWH) to the faculty of the University of South Carolina.

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